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About A LEVEL

MATHEMATICS
Paper 5
(TOPICAL)

period	2010 to 2025
contents	June & November, P5 (S1), Worked Solutions
form	Topic By Topic
compiled for	AS & A Levels
special features	Learning Corner

About **Learning Corner**

This column will further enrich students with the extra information and knowledge provided. It improves the students' ability to solve problems of similar style. Other important guidances are also stated.

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REVISION

- June / November 2025 Paper 5

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Topic 1

Representation of Data - Graphical Representations

Question 1

A library has many identical shelves. All the shelves are full and the numbers of books on each shelf in a certain section are summarised by the following stem-and-leaf diagram.

3	3 6 9 9	(4)
4	6 7	(2)
5	0 1 2 2	(4)
6	0 0 1 1 2 3 4 4 4 4 4 5 5 6 6 6 7 8 8 9	(20)
7	1 1 3 3 3 5 6 6 7 8 9 9	(12)
8	0 2 4 5 5 6 8	(7)
9	0 0 1 2 4 4 4 4 5 5 6 7 7 8 8 9 9 9	(18)

Key: 3 | 6 represents 36 books

(i) Find the number of shelves in this section of the library. [1]

(ii) Draw a box-and-whisker plot to represent the data. [5]

In another section all the shelves are full and the numbers of books on each shelf are summarised by the following stem-and-leaf diagram.

2	1 2 2 2 3 3 4 5 6 6 6 7 9	(13)
3	0 1 1 2 3 4 4 5 6 6 7 7 7 8 8	(15)
4	2 2 3 5 7 7 8 9	(8)

Key: 3 | 6 represents 36 books

(iii) There are fewer books in this section than in the previous section. State one other difference between the books in this section and the books in the previous section. [1]

[N09/P6/Q4]

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Suggested Solution:

(i) Number of shelves = 4 + 2 + 4 + 20 + 12 + 7 + 18 = 67 **(Ans).**

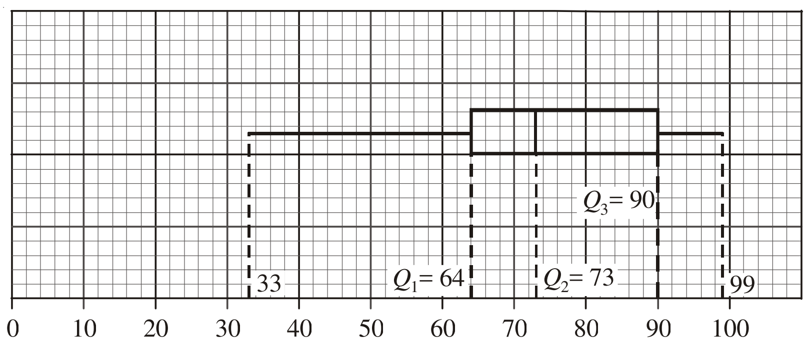
(ii) Lowest number of books in a shelf = 33

Largest number of books in a shelf = 99

Lower quartile, Q_1 = value of $\left(\frac{67+1}{4}\right)^{\text{th}}$ term $= 17^{\text{th}}$ term = 64 **(Ans).**

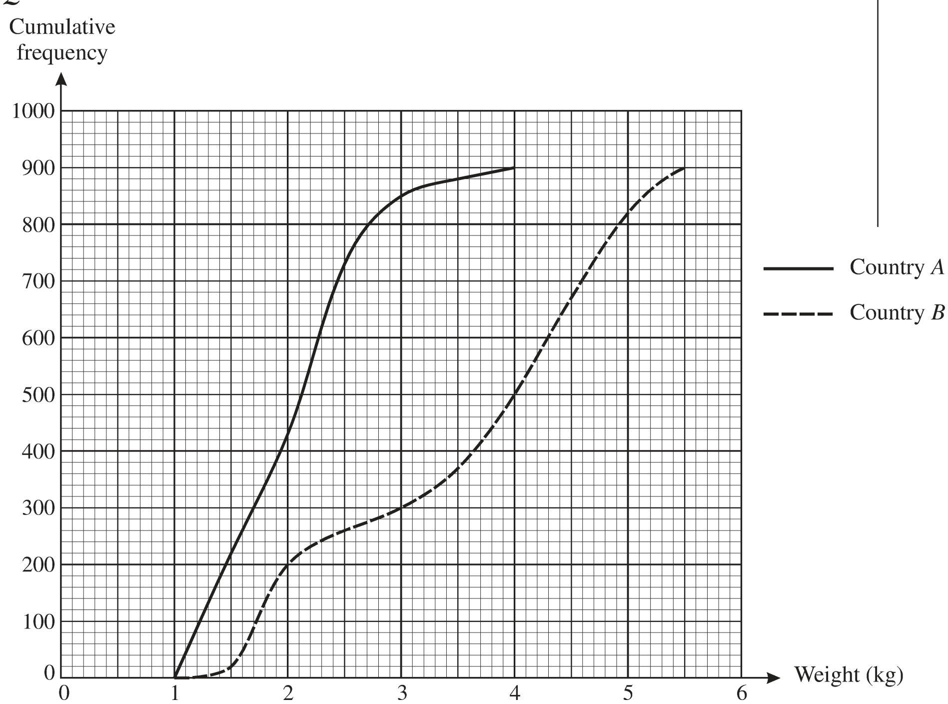
Median, Q_2 = value of $\frac{1}{2}(67+1)^{\text{th}}$ term $= 34^{\text{th}}$ term = 73 **(Ans).**

Upper quartile, Q_3 = value of $\frac{3}{4}(67+1)^{\text{th}}$ term $= 51^{\text{th}}$ term = 90 **(Ans).**



(iii) Books in the second section are thicker or contained more pages. **(Ans).**

Question 2



The birth weights of random samples of 900 babies born in country *A* and 900 babies born in country *B* are illustrated in the cumulative frequency graphs. Use suitable data from these graphs to compare the central tendency and spread of the birth weights of the two sets of babies. [6]

[J10/P6/Q3]

Suggested Solution:

Central Tendency:

From graph,

Median value of birth weights of country $A = 2.05$ kg

Median value of birth weights of country $B = 3.87$ kg

$\Rightarrow$ Average weight of babies in country B is more than in country A.

Spread:

Lower quartile value of birth weight of country $A = 1.50$

Upper quartile value of birth weight of country $A = 2.38$

Interquartile range for country $A = 2.38 - 1.50 = 0.88$ kg

Lower quartile value of birth weight of country $B = 2.14$

Upper quartile value of birth weight of country $B = 4.50$

Interquartile range for country $B = 4.50 - 2.14 = 2.36$ kg

$\Rightarrow$ Country B has a greater spread of weights.

Question 3

The weights in kilograms of 11 bags of sugar and 7 bags of flour are as follows.

Sugar: 1.961 1.983 2.008 2.014 1.968 1.994 2.011 2.017 1.977 1.984 1.989

Flour: 1.945 1.962 1.949 1.977 1.964 1.941 1.953

(i) Represent this information on a back-to-back stem-and-leaf diagram with sugar on the left-hand side. [4]

(ii) Find the median and interquartile range of the weights of the bags of sugar. [3]

[N10/P6/Q4]

Suggested Solution:

(i)

	Sugar			Flour		
			194	1	5	9
			195	3		
	8	1	196	2	4	
		7	197	7		
9	4	3	198			
		4	199			
		8	200			
7	4	1	201			

Key:

1 | 196 | 2

means 1.962 kg for flour
and 1.961 kg for sugar

(ii) Median weight of sugar bags = value of $\left(\frac{1+11}{2}\right)^{\text{th}}$ item

$= \text{value of } 6^{\text{th}} \text{ term} = 1.989$ kg **(Ans).**

Lower quartile = value of 3^{rd} term $= 1.977$ kg

Upper quartile = value of 9^{th} term $= 2.011$ kg

Interquartile Range $= 2.011 - 1.977 = 0.034$ kg **(Ans).**

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Question 4

A hotel has 90 rooms. The table summarises information about the number of rooms occupied each day for a period of 200 days.

Number of rooms occupied	1 – 20	21 – 40	41 – 50	51 – 60	61 – 70	71 – 90
Frequency	10	32	62	50	28	18

(i) Draw a cumulative frequency graph on graph paper to illustrate this information. [4]

(ii) Estimate the number of days when over 30 rooms were occupied. [2]

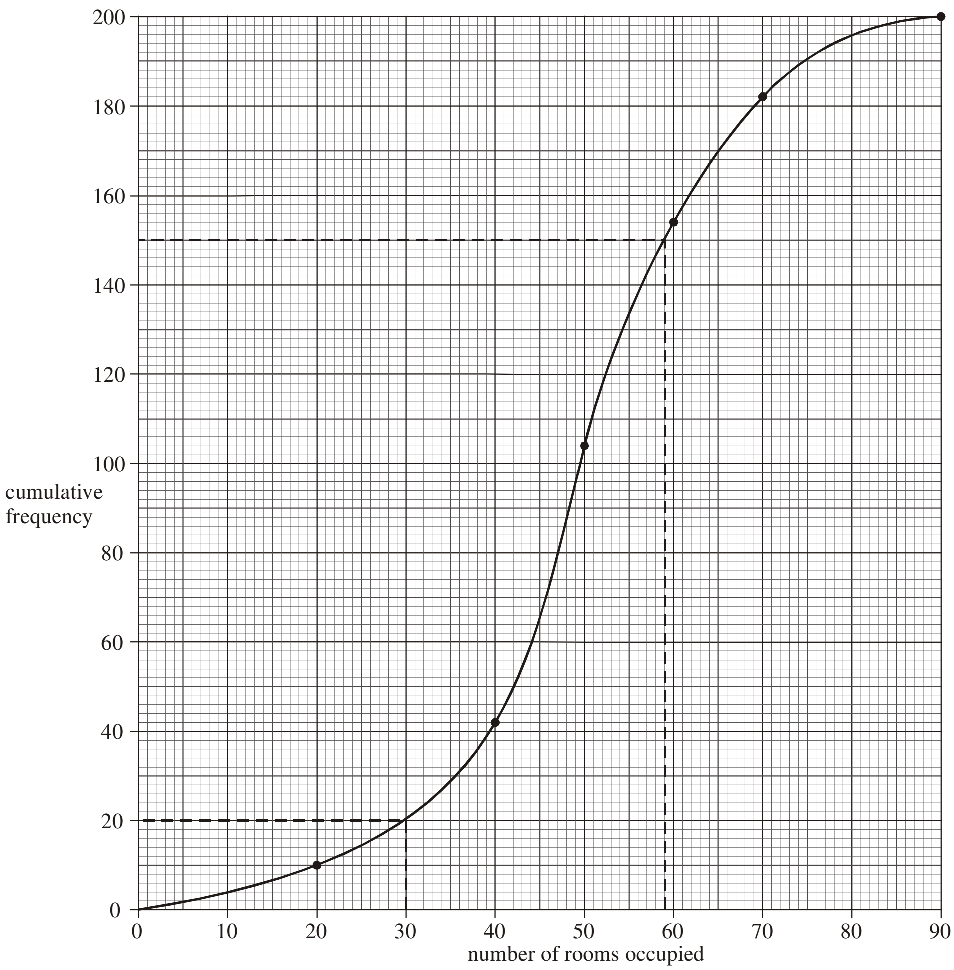
(iii) On 75% of the days at most n rooms were occupied. Estimate the value of n. [2]

[J11/P6/Q5]

Suggested Solution:

(i)

No. of rooms occupied	Frequency	Cumulative frequency (f)
1 – 20	10	10
21 – 40	32	42
41 – 50	62	104
51 – 60	50	154
61 – 70	28	182
71 – 90	18	200



(ii) Number of days = 200 – 20 = 180 **(Ans).**

(iii) $\frac{75}{100} \times 200 = 150$

$\therefore$ from graph, $n = 59$ rooms. **(Ans).**

Question 5

The weights of 220 sausages are summarised in the following table.

Weight (grams)	<20	<30	<40	<45	<50	<60	<70
Cumulative frequency	0	20	50	100	160	210	220

(i) State which interval the median weight lies in. [1]

(ii) Find the smallest possible value and the largest possible value for the interquartile range. [2]

(iii) State how many sausages weighed between 50 g and 60 g. [1]

(iv) On graph paper, draw a histogram to represent the weights of the sausages. [4]

[N11/P6/Q4]

Suggested Solution:

(i) Median weight lies in the interval: $45 < x < 50$. **(Ans).**

(ii) Lower quartile lies in the interval: $40 < x < 45$

Upper quartile lies in the interval: $50 < x < 60$

$\Rightarrow$ smallest possible value of interquartile range = 5

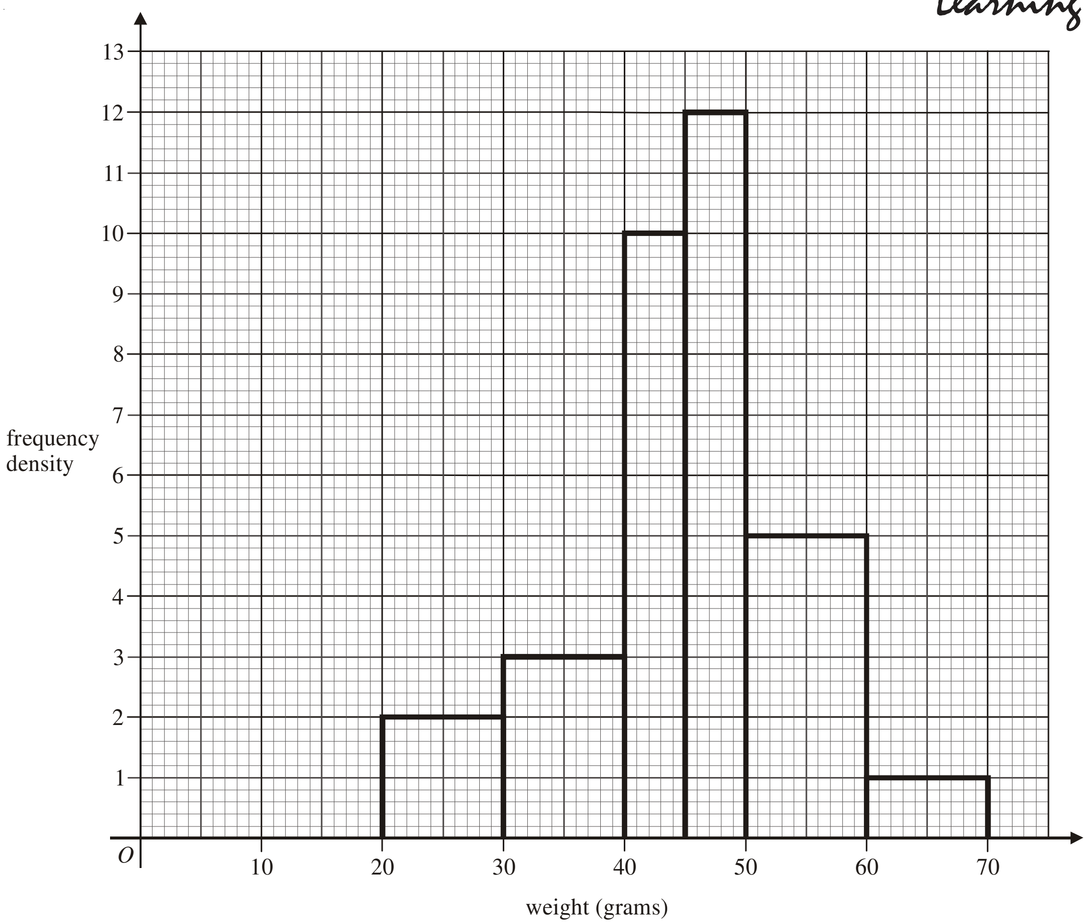
largest possible value of interquartile range = 20

(iii) 50

(iv)

Class interval	Class width	Cumulative frequency	Frequency	Frequency density
$0 \le x < 20$	20	0	0	0
$20 \le x < 30$	10	20	$20 - 0 = 20$	2
$30 \le x < 40$	10	50	$50 - 20 = 30$	3
$40 \le x < 45$	5	100	$100 - 50 = 50$	10
$45 \le x < 50$	5	160	$160 - 100 = 60$	12
$50 \le x < 60$	10	210	$210 - 160 = 50$	5
$60 \le x < 70$	10	220	$220 - 210 = 10$	1

Frequency density $= \frac{\text{frequency}}{\text{class width}}$



Question 6

The back-to-back stem-and-leaf diagram shows the values taken by two variables A and B.

	A		*B*	
(3)	3 1 0	15	1 3 3 5	(4)
(2)	4 1	16	2 2 3 4 4 5 7 7 7 8	(10)
(3)	8 3 3	17	0 1 3 3 3 4 6 6 7 9 9	(11)
(12)	9 8 8 6 5 5 4 3 2 1 1 0	18	2 4 7	(3)
(8)	9 9 8 8 6 5 4 2	19	1 5	(2)
(5)	9 8 7 1 0	20	4	(1)

Key: 4 | 16 | 7 means $A = 0.164$ and $B = 0.167$.

(i) Find the median and the interquartile range for variable A. [3]

(ii) You are given that, for variable B, the median is 0.171, the upper quartile is 0.179 and the lower quartile is 0.164. Draw box-and-whisker plots for A and B in a single diagram on graph paper. [3]

[J12/P6/Q4]

Suggested Solution:

(i) Median of variable A = value of $\left(\frac{33+1}{2}\right)^{\text{th}}$ term

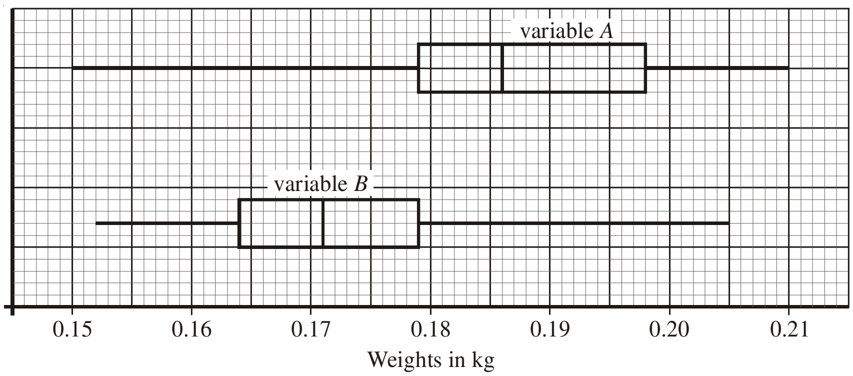
$= \text{value of } 17^{\text{th}} \text{ term} = 0.186$ **(Ans).**

lower quartile $= \left(\frac{0.180+0.178}{2}\right) = 0.179$

upper quartile $= 0.198$

$\therefore$ interquartile range $= 0.198 - 0.179 = 0.019$ **(Ans).**

(ii)



Question 7

The table summarises the times that 112 people took to travel to work on a particular day.

Time to travel to work (t minutes)	$0 < t \le 10$	$10 < t \le 15$	$15 < t \le 20$	$20 < t \le 25$	$25 < t \le 40$	$40 < t \le 60$
Frequency	19	12	28	22	18	13

(i) State which time interval in the table contains the median and which time interval contains the upper quartile. [2]

(ii) On graph paper, draw a histogram to represent the data. [4]

(iii) Calculate an estimate of the mean time to travel to work. [2]

[N12/P6/Q3]

Suggested Solution:

(i) Median = value of $\frac{1}{2}(112)^{\text{th}}$ term $= 56^{\text{th}}$ term

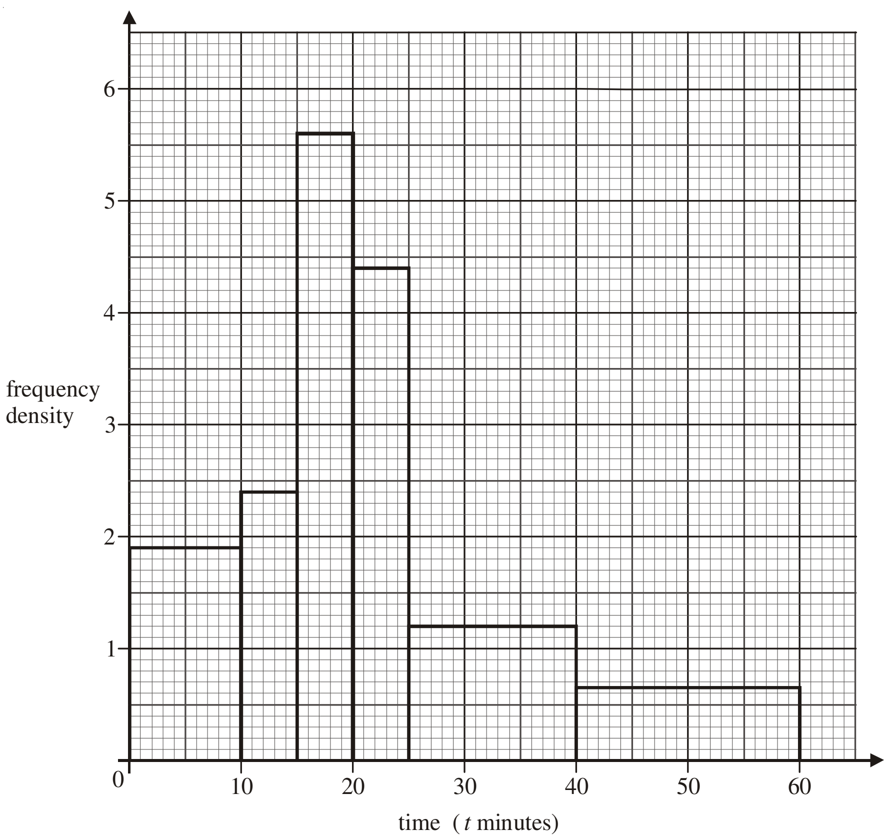
$\Rightarrow$ median lies in the interval $15 < t \le 20$. **(Ans).**

Upper quartile = value of $\frac{3}{4}(112)^{\text{th}}$ term $= 84^{\text{th}}$ value

$\Rightarrow$ upper quartile lies in the interval $25 < t \le 40$ **(Ans).**

(ii)

Time to travel to work (t minutes)	Frequency (f)	class width	Frequency density	Mid value (t)
$0 < t \le 10$	19	10	$=\frac{19}{10}=1.9$	5
$10 < t \le 15$	12	5	$=\frac{12}{5}=2.4$	12.5
$15 < t \le 20$	28	5	$=\frac{28}{5}=5.6$	17.5
$20 < t \le 25$	22	5	$=\frac{22}{5}=4.4$	22.5
$25 < t \le 40$	18	15	$=\frac{18}{15}=1.2$	32.5
$40 < t \le 60$	13	20	$=\frac{13}{20}=0.65$	50
	$\sum f = 112$			



(iii) Mean time, $\bar{t} = \frac{\sum ft}{\sum f} = \frac{(19\times5)+(12\times12.5)+(28\times17.5)+(22\times22.5)+(18\times32.5)+(13\times50)}{112}$

$= \frac{2465}{112} = 22.0$ minutes **(Ans).**

Question 8

The following are the annual amounts of money spent on clothes, to the nearest \$10, by 27 people.

10	40	60	80	100	130	140	140	140
150	150	150	160	160	160	160	170	180
180	200	210	250	270	280	310	450	570

(i) Construct a stem-and-leaf diagram for the data. [3]

(ii) Find the median and the interquartile range of the data. [3]

An 'outlier' is defined as any data value which is more than 1.5 times the interquartile range above the upper quartile, or more than 1.5 times the interquartile range below the lower quartile.

(iii) List the outliers. [3]

[J13/P6/Q5]

Suggested Solution:

(i)

stem	leaf
0	1 4 6 8
1	0 3 4 4 4 5 5 5 6 6 6 6 7 8 8
2	0 1 5 7 8
3	0
4	5
5	7

Key: 2 | 5 means \$250

(ii) Median, $Q_2 = \text{value of } \left(\frac{n+1}{2}\right)^{\text{th}} \text{ term} = \text{value of } 14^{\text{th}} \text{ term} = 160$ **(Ans).**

Lower quartile, $Q_1 = \text{value of} \left(\frac{n+1}{4}\right)^{\text{th}} \text{ term} = \text{value of } 7^{\text{th}} \text{ term} = 140$

Upper quartile, $Q_3 = \text{value of } \frac{3}{4}(n+1)^{\text{th}} = \text{value of } 21^{\text{st}} \text{ term} = 210$

Interquartile range $= Q_3 - Q_1 = 210 - 140 = 70$ **(Ans).**

(iii) Outliers are less than $Q_1 - 1.5(\text{IQR})$ or more than $Q_3 + (1.5)(\text{IQR})$

$\Rightarrow$ outliers < 35 or outliers > 315

$\therefore$ outliers $= 10, 450, 570$ **(Ans).**

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Question 23

Two machines, *A* and *B*, produce metal rods of a certain type. The lengths, in metres, of 19 rods produced by machine *A* and 19 rods produced by machine *B* are shown in the following back-to-back stem-and-leaf diagram.

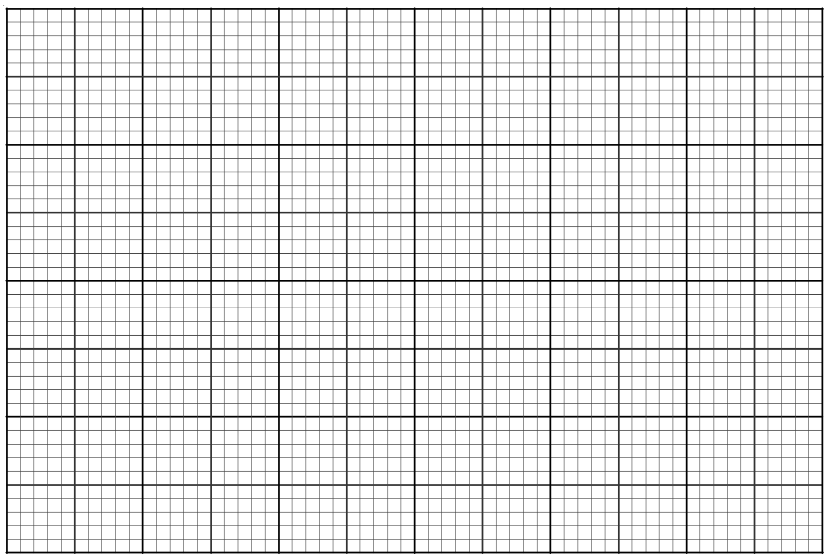
			A						*B*			
						21	1	2	4			
		7	6	3	0	22	2	4	5	5	6	
8	7	4	3	1	1	23	0	2	6	8	9	9
	5	5	5	3	2	24	3	3	4	6		
		4	3	1	0	25	6					

Key: 7 |22| 4 means 0.227 m for machine *A* and 0.224 m for machine *B*.

(a) Find the median and the interquartile range for machine *A*. [3]

It is given that for machine *B* the median is 0.232 m, the lower quartile is 0.224 m and the upper quartile is 0.243 m.

(b) Draw box-and-whisker plots for *A* and *B*. [3]



(c) Hence make two comparisons between the lengths of the rods produced by machine *A* and those produced by machine *B*. [2]

[J20/P5/Q3]

Suggested Solution:

(a) Median = value of $\left(\dfrac{19+1}{2}\right)^{\text{th}}$ term $= 0.238$

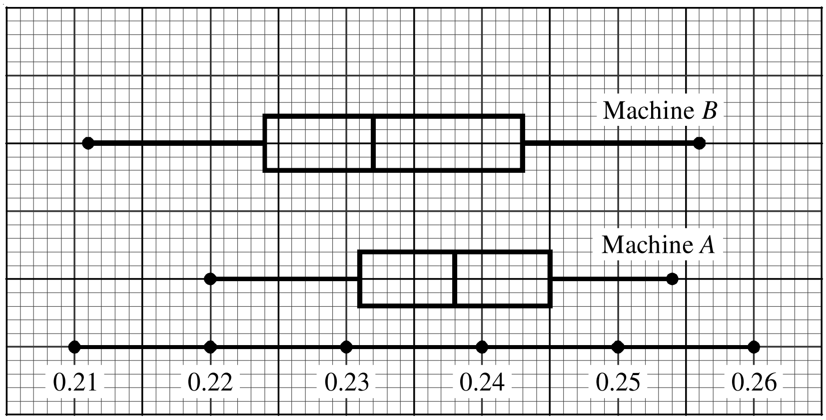
Lower Quartile, Q_1 = value of $\left(\dfrac{19+1}{4}\right)^{\text{th}}$ term $= 0.231$

Upper Quartile, Q_3 = value of $3\left(\dfrac{19+1}{4}\right)^{\text{th}}$ term $= 0.245$

Interquartile range $= Q_3 - Q_1$

$= 0.245 - 0.231 = 0.014$ **(Ans).**

(b)



(c) 1. Spread in lengths of rods produced by machine *B* is more than machine *A*.

2. Rods produced by machine *A* are longer than machine *B*.

Question 24

The following table gives the weekly snowfall, in centimetres, for 11 weeks in 2018 at two ski resorts, Dados and Linva.

Dados	6	8	12	15	10	36	42	28	10	22	16
Linva	2	11	15	16	0	32	36	40	10	12	9

(a) Represent the information in a back-to-back stem-and-leaf diagram. [4]

(b) Find the median and the interquartile range for the weekly snowfall in Dados. [3]

(c) The median, lower quartile and upper quartile of the weekly snowfall for Linva are 12, 9 and 32 cm respectively. Use this information and your answers to part **(b)** to compare the central tendency and the spread of the weekly snowfall in Dados and Linva. [2]

[N20/P5/Q5]

Suggested Solution:

(a)

Linva						Dados				
		9	2	0	0	6	8			
6	5	2	1	0	1	0	0	2	5	6
					2	2	8			
			6	2	3	6				
				0	4	2				

Key

2 | 3 | 6

means 36 cm snowfall in Dados
and 32 cm snowfall in Linva

(b) Median = value of 6th term = 15 **(Ans).**

Lower Quartile = value of 3rd term = 10

Upper Quartile = value of 9th term = 28

$\therefore$ Interquartile Range = $Q_3 - Q_1$

$= 28 - 10 = 18$ **(Ans).**

(c) Interquartile range for snowfall in Linva = $32 - 9 = 23$

Median for snowfall in Linva = 12

Since median value in Dados is higher than Linva

Therefore average weekly snowfall in Dados is greater.

Interquartile range in Linva is more than Dados

Therefore spread in weekly snowfall in Linva is higher.

Question 25

The heights, in cm, of the 11 basketball players in each of two clubs, the Amazons and the Giants, are shown below.

Amazons	205	198	181	182	190	215	201	178	202	196	184
Giants	175	182	184	187	189	192	193	195	195	195	204

(a) State an advantage of using a stem-and-leaf diagram compared to a box-and-whisker plot to illustrate this information. [1]

(b) Represent the data by drawing a back-to-back stem-and-leaf diagram with Amazons on the left-hand side of the diagram. [4]

(c) Find the interquartile range of the heights of the players in the Amazons. [2]

Four new players join the Amazons. The mean height of the 15 players in the Amazons is now 191.2 cm. The heights of three of the new players are 180 cm, 185 cm and 190 cm.

(d) Find the height of the fourth new player. [3]

[J21/P5/Q7]

Suggested Solution:

(a) In stem-and-leaf diagram all data is included.

(b)

Amazons				Giants				
		8	17	5				
4	2	1	18	2	4	7	9	
8	6	0	19	2	3	5	5	5
5	2	1	20	4				
		5	21					

Key

0 | 19 | 2

means 190 cm for Amazons and 192 cm for Giants

(c) Lower Quartile = 182, Upper Quartile = 202

$\therefore$ Interquartile Range = $202 - 182 = 20$ cm **(Ans).**

(d) Mean height of 15 players, $\overline{x} = 191.2$

Sum of heights of 15 players $= \sum x_{15} = 191.2 \times 15 = 2868$ .... (1)

Let x be the height of fourth new player.

Sum of height of 11 Amazons players

$= 205 + 198 + 181 + 182 + 190 + 215 + 201 + 178 + 202 + 196 + 184 = 2132$

Sum of heights of 11 players and 4 new players

$= 2132 + 180 + 185 + 190 + x = 2687 + x$ .... (2)

From (1) & (2), $2687 + x = 2868 \quad \Rightarrow \quad x = 181$ cm **(Ans).**

Question 26

The distances, x m, travelled to school by 140 children were recorded. The results are summarised in the table below.

Distance, x m	$x \le 200$	$x \le 300$	$x \le 500$	$x \le 900$	$x \le 1200$	$x \le 1600$
Cumulative frequency	16	46	88	122	134	140

(a) On the grid, draw a cumulative frequency graph to represent these results. [2]

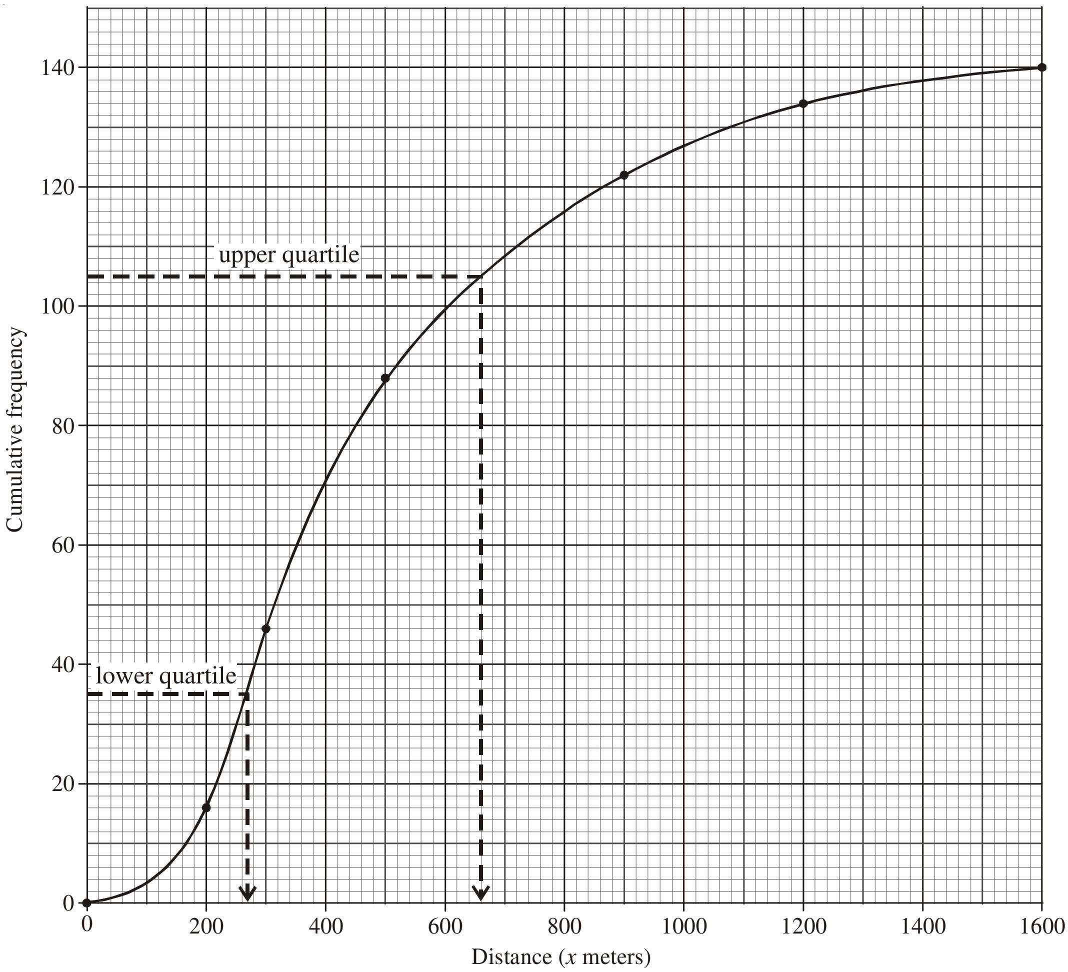
(b) Use your graph to estimate the interquartile range of the distances. [2]

(c) Calculate estimates of the mean and standard deviation of the distances. [6]

[N21/P5/Q7]

Suggested Solution:

(a)



(b) From graph, Lower Quartile = 270, Upper Quartile = 660

$\therefore$ Interquartile range $= 660 - 270 = 390$ m **(Ans).**

(c)

Distance (x m)	Mid value (x)	Cumulative frequency	Frequency (f)	fx	fx^2
$0 < x \le 200$	100	16	16	1600	160000
$200 < x \le 300$	250	46	30	7500	1875000
$300 < x \le 500$	400	88	42	16800	6720000
$500 < x \le 900$	700	122	34	23800	16660000
$900 < x \le 1200$	1050	134	12	12600	13230000
$1200 < x \le 1600$	1400	140	6	8400	11760000
				70700	50405000

Mean, $\bar{x} = \dfrac{\sum fx}{\sum f} = \dfrac{70700}{140} = 505$ **(Ans).**

$$\text{Standard Deviation} = \sqrt{\frac{\sum fx^2}{\sum f} - (\bar{x})^2}$$

$$= \sqrt{\frac{50405000}{140} - (505)^2} = 324 \quad \textbf{(Ans).}$$

Question 27

The back-to-back stem-and-leaf diagram shows the diameters, in cm, of 19 cylindrical pipes produced by each of two companies, *A* and *B*.

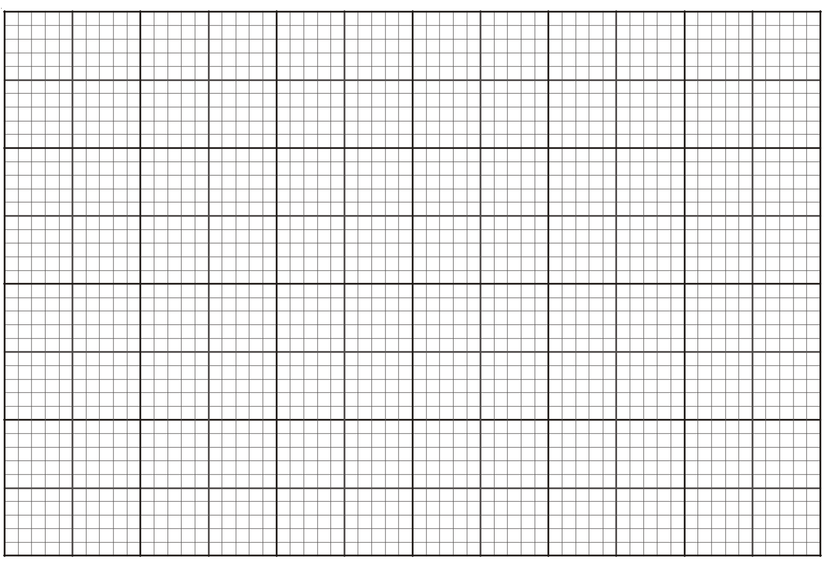
Company *A*							Company *B*				
					4	33	1	2	8		
	9	8	3	2	0	34	1	6	8	9	9
8	7	5	4	1	1	35	1	2	2	3	
		9	6	5	2	36	5	6			
			4	3	1	37	0	3	4		
						38	2	8			

Key: 1 | 35 | 3 means the pipe diameter from company *A* is 0.351 cm and from company *B* is 0.353 cm.

(a) Find the median and interquartile range of the pipes produced by company *A*. [3]

It is given that for the pipes produced by company *B* the lower quartile, median and upper quartile are 0.346 cm, 0.352 cm and 0.370 cm respectively.

(b) Draw box-and-whisker plots for companies *A* and *B* on the grid below. [3]



(c) Make one comparison between the diameters of the pipes produced by companies *A* and *B*. [1]

[J22/P5/Q3]

Learning CORNER

Suggested Solution:

(a) Median, $Q_2 = \text{value of } \left(\frac{n+1}{2}\right)^{\text{th}}$ term = value of 10th term = 0.355 **(Ans).**

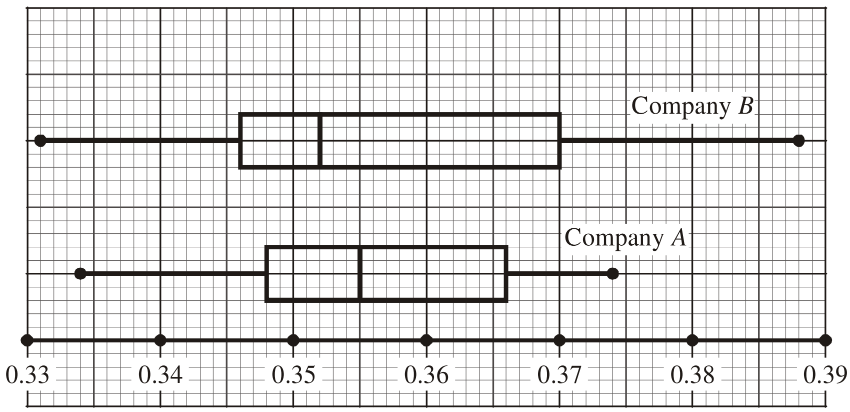
Lower Quartile, $Q_1 = \text{value of } \left(\frac{n+1}{4}\right)^{\text{th}}$ term = value of 5th term = 0.348

Upper Quartile, $Q_3 = \text{value of } 3\left(\frac{n+1}{4}\right)^{\text{th}}$ term = value of 15th term = 0.366

$\therefore$ Interquartile Range $= Q_3 - Q_1$

$= 0.018$ **(Ans).**

(b)



(c) Spread is more in diameters of pipes made by company *B* than company *A*.
Or
Diameters of pipes, made by company *A*, are bigger than company *B*.

Question 28

The times taken, in minutes, to complete a word processing task by 250 employees at a particular company are summarised in the table.

Time taken (t minutes)	$0 \le t < 20$	$20 \le t < 40$	$40 \le t < 50$	$50 \le t < 60$	$60 \le t < 100$
Frequency	32	46	96	52	24

(a) Draw a histogram to represent this information. [4]

From the data, the estimate of the mean time taken by these 250 employees is 43.2 minutes.

(b) Calculate an estimate for the standard deviation of these times. [3]

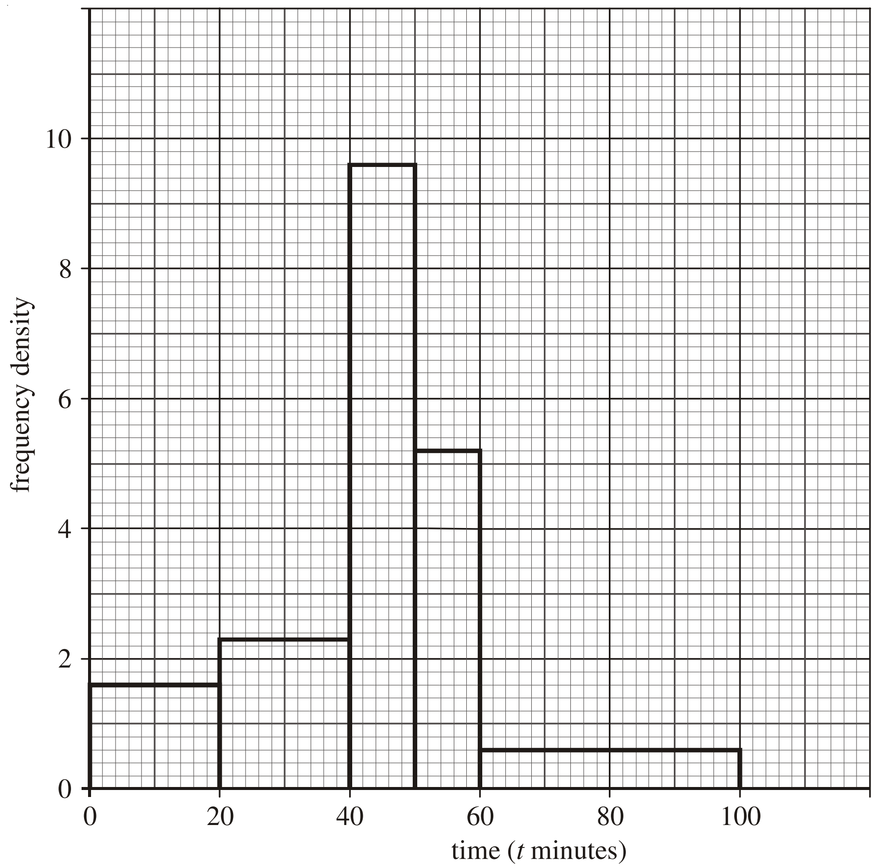
[N22/P5/Q4]

Suggested Solution:

(a)

Time taken (t minutes)	Class width	Frequency	Frequency density
$0 \leq t < 20$	20	32	1.6
$20 \leq t < 40$	20	46	2.3
$40 \leq t < 50$	10	96	9.6
$50 \leq t < 60$	10	52	5.2
$60 \leq t < 100$	40	24	0.6

Learning CORNER



(b)

Time (t minutes)	Mid value (t)	Frequency f	t^2	ft^2
$0 \le t < 20$	10	32	100	3200
$20 \le t < 40$	30	46	900	41 400
$40 \le t < 50$	45	96	2025	194 400
$50 \le t < 60$	55	52	3025	157 300
$60 \le t < 100$	80	24	6400	153 600
		$\sum f = 250$		$\sum ft^2 = 549900$

Mean, $\bar{x} = 43.2$ minutes

$$\text{Standard Deviation} = \sqrt{\frac{\sum ft^2}{\sum f} - (\bar{x})^2}$$

$$= \sqrt{\frac{549\,900}{250} - (43.2)^2}$$

$$= 18.258 \approx 18.3 \quad \textbf{(Ans).}$$

Question 29

The following back-to-back stem-and-leaf diagram represents the monthly salaries, in dollars, of 27 employees at each of two companies, A and B.

Company A		Company B
5 4 1 1 0	25	4 4 5 6 6 7
9 9 8 7 2 1 0	26	0 1 3 5 5 7 9 9
8 6 4 2 1 0	27	1 3 4 6 6 8 8
6 5 4 2 0	28	0 1 2 2 2
9 8 5	29	
1	30	9

Key: 1 | 27 | 6 means \$2710 for company A and \$2760 for company B

(a) Find the median and the interquartile range of the monthly salaries of employees in company A. [3]

The lower quartile, median and upper quartile for company B are \$2600, \$2690 and \$2780 respectively.

(b) Draw two box-and-whisker plots in a single diagram to represent the information for the salaries of employees at companies A and B. [3]

(c) Comment on whether the mean would be a more appropriate measure than the median for comparing the given information for the two companies. [1]

[J23/P5/Q3]

Suggested Solution:

(a) Median, $Q_2 = \text{value of } \left(\dfrac{27+1}{2}\right)^{\text{th}} \text{ term} = \text{value of 14th term} = \$2710$ (**Ans**).

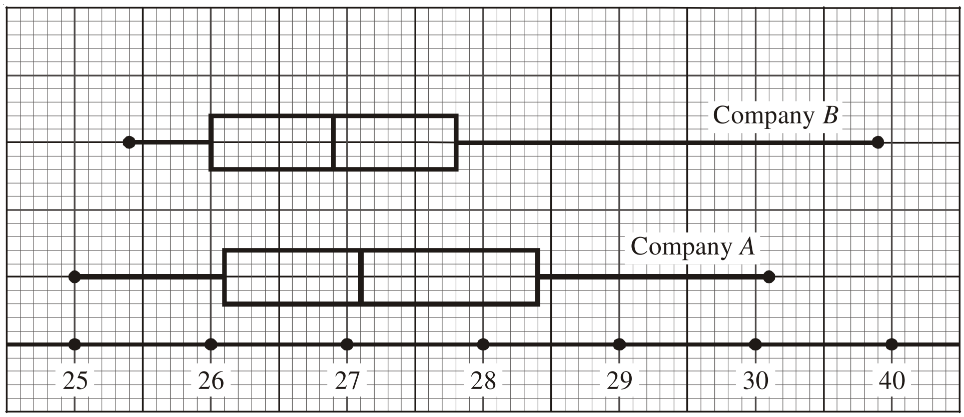
Lower Quartile, $Q_1 = \text{value of 7th term} = \$2610$

Upper Quartile, $Q_3 = \text{value of 21st term} = \$2840$

Interquartile range $= Q_3 - Q_1$

$= \$2840 - \$2610 = \$230$ (**Ans**).

(b)



Salaries ($00)

(c) Median is a better measure of central tendency than mean because in company *B* salary of one employee is significantly large i.e. $3090 that effects the value of the mean.

Question 30

The heights, in cm, of the 11 players in each of two teams, the Aces and the Jets, are shown in the following table.

Aces	180	174	169	182	181	166	173	182	168	171	164
Jets	175	174	188	168	166	174	181	181	170	188	190

(a) Draw a back-to-back stem-and-leaf diagram to represent this information with the Aces on the left-hand side of the diagram. [4]

(b) Find the median and the interquartile range of the heights of the players in the Aces. [3]

(c) Give one comment comparing the spread of the heights of the Aces with the spread of the heights of the Jets. [1]

[N23/P5/Q4]

Suggested Solution:

(a)

Aces					Jets			
9	8	6	4	16	6	8		
	4	3	1	17	0	4	4	5
2	2	1	0	18	1	1	8	8
				19	0			

Key

0 | 18 | 1 means 180 cm is height of Ace's player
and 181 cm is height of Jet's player

(b) Lower Quartile, $Q_1 = \text{Value of } \left(\frac{11+1}{4}\right)^{\text{th}} \text{ term}$

$= \text{Value of } 3^{\text{rd}} \text{ term} = 168 \text{ cm}$

Median, $Q_2 = \text{Value of } \left(\frac{11+1}{2}\right)^{\text{th}} \text{ term}$

$= \text{Value of } 6^{\text{th}} \text{ term} = 173 \text{ cm}$ (**Ans**).

Upper Quartile, $Q_3 = \text{Value of } \frac{3}{4}(11+1)^{\text{th}} \text{ term}$

$= \text{Value of } 9^{\text{th}} \text{ term} = 181 \text{ cm}$

Interquartile range $= Q_3 - Q_1$

$= 181 - 168 = 13 \text{ cm}$ (**Ans**).

(c) Spread in Jet's players heights is more than Ace's players heights.

Question 31

The back-to-back stem-and-leaf diagram shows the annual salaries of 19 employees at each of two companies, Petral and Ravon.

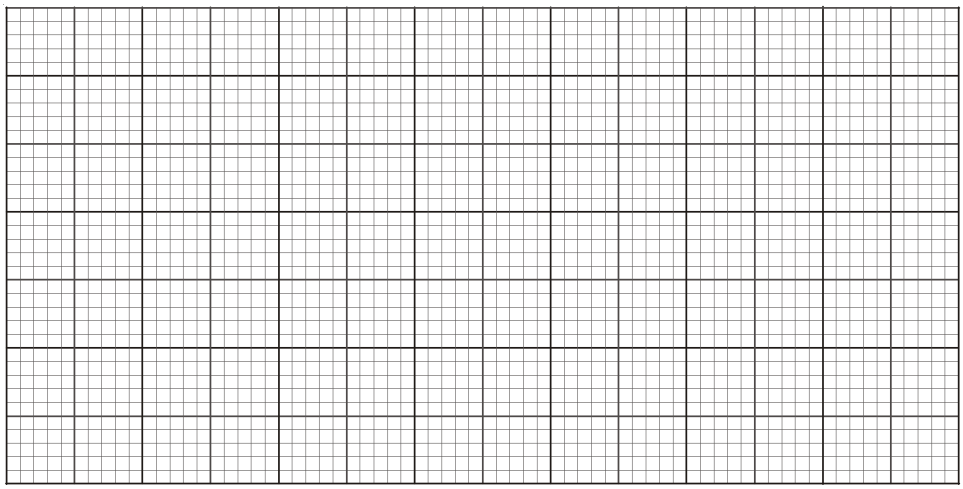
Petral		Ravon
3 0 0	30	2 6
9 9 8 2 2 1	31	1 5
5 5 4 0	32	0 0 2
7 5 3	33	0 4 8 9
1 0	34	1 1 3 4 6
	35	3
8	36	7 9

Key: 2 | 31 | 5 means \$31 200 for a Petral employee and \$31 500 for a Ravon employee.

(a) Find the median and the interquartile range of the salaries of the Petral employees. [3]

The median salary of the Ravon employees is \$33 800, the lower quartile is \$32 000 and the upper quartile is \$34 400.

(b) Represent the data shown in the back-to-back stem-and-leaf diagram by a pair of box-and-whisker plots in a single diagram. [3]



(c) Comment on whether the mean or the median would be a better representation of the data for the employees at Petral. [1]

[J24/P5/Q4]

Suggested Solution:

(a) Lower quartile, $Q_1 = \text{value of}\left(\frac{19+1}{4}\right)^{\text{th}}$ term

$= \text{value of 5th term} = \$31200$

Upper quartile, $Q_3 = \text{value of}\ \frac{3}{4}(19+1)^{\text{th}}$ term

$= \text{value of 15th term} = \$33500$

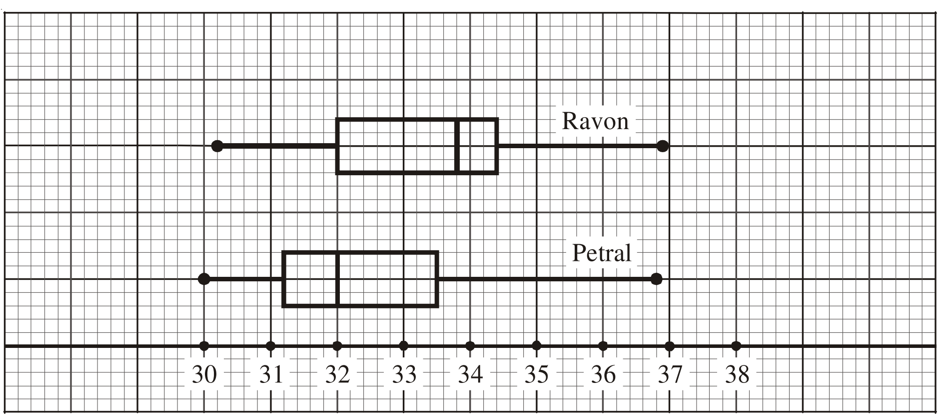
$\therefore$ Interquartile range $= Q_3 - Q_1$

$= \$33500 - \$31200 = \$2300$ **(Ans).**

Median, $Q_2 = \text{value of}\left(\frac{19+1}{2}\right)^{\text{th}}$ term

$= \text{value of 10th term} = \$32000$ **(Ans).**

(b)



Salaries ($000)

(c) Median would be a better representation for employees at Petral, because there is a significantly large value of $36 800 in the data which effects the value of mean.

Learning CORNER

Topic 8
The Normal Distribution

Question 1

The lengths of new pencils are normally distributed with mean 11 cm and standard deviation 0.095 cm.

(i) Find the probability that a pencil chosen at random has a length greater than 10.9 cm. [2]

(ii) Find the probability that, in a random sample of 6 pencils, at least two have lengths less than 10.9 cm. [3]

[J10/P6/Q2]

Suggested Solution:

(i) Let X be the length of pencil which is a function of normal distribution

i.e. $X \sim N(11,\ 0.095^2)$

$P(X > 10.9)$

$\Rightarrow\ P\left(\frac{X-\mu}{\sigma} > \frac{10.9-11}{0.095}\right)$ standardizing X using $\frac{X-\mu}{\sigma} = Z$

$= P(Z > -\frac{0.1}{0.095})$

$= P(Z > -1.053)$

$= 1 - \Phi(-1.053)$ $\because\ P(Z > a) = 1 - \Phi(a)$

$= 1 - [1 - \Phi(1.053)]$ $\because\ \Phi(-a) = 1 - \Phi(a)$

$= \Phi(1.053)$

$=\ 0.8538 \approx 0.854$ (3 sf)

$\therefore$ P(a pencil chosen has a length greater than 10.9cm) $= 0.854$ **(Ans).**

(ii) No. of pencils $= n = 6$

P(length is less than 10.9 cm) $= p = 1 - 0.8538 = 0.1462$

P(length is not less than 10.9 cm) $= q = 0.8538$

Let X be the function of Binomial distribution, i.e. $X \sim B(6,\ 0.1462)$

$P(X \geq 2) = P(X=2) + P(X=3) + P(X=4) + P(X=5) + P(X=6)$

$= 1 - [\mathrm{P}(X=0) + \mathrm{P}(X=1)]$

$= 1 - \left[{}^6C_0(0.1462)^0(0.8538)^6 + {}^6C_1(0.1462)^1(0.8538)^5\right]$

$= 1 - (0.3874 + 0.3980)$

$=\ 0.2146 \approx 0.215$ **(Ans).**

Note that:

$X \sim \text{Bin}\ (n,\ p)$

n = Total no. of trials

p = Prob. of success

q = Prob. of failure

$P(X = x) = {}^nC_x\, p^x\, q^{n-x}$

Question 2

The random variable X is normally distributed with mean μ and standard deviation σ.

(i) Given that $5\sigma = 3\mu$, find $\mathrm{P}(X < 2\mu)$. [3]

(ii) With a different relationship between μ and σ, it is given that $\mathrm{P}(X < \frac{1}{3}\mu) = 0.8524$. Express μ in terms of σ. [3]

[J10/P6/Q4]

Learning CORNER

Suggested Solution:

(i) $P(X < 2\mu)$ $\qquad X \sim (\mu, \sigma^2)$

$\Rightarrow P\left(\frac{X-\mu}{\sigma} < \frac{2\mu-\mu}{\sigma}\right)$ $\qquad$ standardizing X using $\frac{X-\mu}{\sigma} = Z$

$\Rightarrow P\left(Z < \frac{\mu}{\sigma}\right)$ $\qquad \because Z \sim N(0, 1)$

$\Rightarrow P\left(Z < \frac{\frac{5}{3}\sigma}{\sigma}\right) \Rightarrow P(Z < 1.667) \Rightarrow \Phi(1.667) = 0.952$ **(Ans).**

(ii) $P(X < \frac{1}{3}\mu) = 0.8524$ $\qquad X \sim (\mu, \sigma^2)$

$\Rightarrow P\left(\frac{X-\mu}{\sigma} < \frac{\frac{\mu}{3}-\mu}{\sigma}\right) = 0.8524$

$\Rightarrow P\left(Z < -\frac{2\mu}{3\sigma}\right) = 0.8524$

Let $a = -\frac{2\mu}{3\sigma}$

$\Rightarrow P(Z < a) = 0.8524$ $\qquad \because a$ is positive

$\Phi(a) = \Phi(1.047)$

$a = 1.047$

$\Rightarrow -\frac{2\mu}{3\sigma} = 1.047 \Rightarrow \mu = -1.57\sigma$ **(Ans).**

Question 3

The distance the Zotoc car can travel on 20 litres of fuel is normally distributed with mean 320 km and standard deviation 21.6 km. The distance the Ganmor car can travel on 20 litres of fuel is normally distributed with mean 350 km and standard deviation 7.5 km. Both cars are filled with 20 litres of fuel and are driven towards a place 367 km away.

(i) For each car, find the probability that it runs out of fuel before it has travelled 367 km. [3]

(ii) The probability that a Zotoc car can travel at least $(320 + d)$ km on 20 litres of fuel is 0.409. Find the value of d. [4]

[N10/P6/Q5]

Suggested Solution:

(i) Given that, $\mu = 320$ km, $\sigma = 21.6$ km

where μ is the mean and σ is the standard deviation

Let X be the distance travelled by Zotoc car, which is a function of normal distribution, i.e. $X \sim N(320, 21.6^2)$

$P(X < 367)$

$= P\left(\frac{X-\mu}{\sigma} < \frac{367-320}{21.6}\right)$ $\qquad$ standardizing X using $\frac{X-\mu}{\sigma} = Z$

$= P\left(Z < \frac{47}{21.6}\right)$ $\qquad \because Z \sim N(0, 1)$

$= P(Z < 2.1759) = \Phi(2.176) = 0.985$ (3sf) **(Ans).**

Candidates should know the parameters of the various distributions and what it represents. For $X \sim N(\mu, \sigma^2)$, μ is the mean and σ is the standard deviation.

Let Y be the distance travelled by Ganmor car, which is a function of normal distribution, i.e. $Y \sim N(350,\ 7.5^2)$

$P(X < 367)$

$= P\left(\dfrac{X-\mu}{\sigma} < \dfrac{367-350}{7.5}\right)$ standardizing X using $\dfrac{X-\mu}{\sigma} = Z$

$= P\left(Z < \dfrac{17}{7.5}\right)$ $\because\ Z \sim N(0,\ 1)$

$= P(Z < 2.267)$

$= \Phi(2.276) = 0.988$ (3sf) **(Ans).**

(ii) Given that,

$$P(X \geq 320 + d) = 0.409$$

$$\Rightarrow\ P\left(\frac{X-\mu}{\sigma} \geq \frac{320+d-320}{21.6}\right) = 0.409$$

$$\Rightarrow\ P\left(Z \geq \frac{d}{21.6}\right) = 0.409 \ \Rightarrow\ 1 - \Phi\left(\frac{d}{21.6}\right) = 0.409$$

$$\Rightarrow\ \Phi\left(\frac{d}{21.6}\right) = 1 - 0.409 \ \Rightarrow\ \Phi\left(\frac{d}{21.6}\right) = 0.591$$

$$\Rightarrow\ \Phi\left(\frac{d}{21.6}\right) = \Phi(0.23) \ \Rightarrow\ \frac{d}{21.6} = 0.23 \ \Rightarrow\ d = 4.968 \approx 4.97$$ **(Ans).**

Question 4

(i) State three conditions that must be satisfied for a situation to be modelled by a binomial distribution. [2]

On any day, there is a probability of 0.3 that Julie's train is late.

(ii) Nine days are chosen at random. Find the probability that Julie's train is late on more than 7 days or fewer than 2 days. [3]

(iii) 90 days are chosen at random. Find the probability that Julie's train is late on more than 35 days or fewer than 27 days. [5]

[N10/P6/Q6]

Suggested Solution:

(i) 1. Number of trials are fixed.

2. Every outcome of each trial is deemed either a success or a failure.

3. Probability, p, of a successful outcome is the same for each trial.

(ii) No. of days $= n = 9$

Probability that train is late $= p = 0.3$

Probability that train is not late $= q = 0.7$

$\Rightarrow\ X \sim B(9, 0.3)$

P(train is late on more than 7 days or fewer than 2 days)

$= P(X > 7) + P(X < 2)$ $\because\ X \sim B(9, 0.3)$

$= P(X = 8) + P(X = 9) + P(X = 0) + P(X = 1)$

$= {}^9C_8(0.3)^8(0.7)^1 + {}^9C_9(0.3)^9(0.7)^0 + {}^9C_0(0.3)^0(0.7)^9 + {}^9C_1(0.3)^1(0.7)^8$

$= 0.0004133 + 0.00001968 + 0.040354 + 0.15565$

$= 0.1964 \approx 0.196$ **(Ans).**

Note that:

$X \sim \text{Bin}\ (n,\ p)$

$n =$ Total no. of trials

$p =$ Prob. of success

$q =$ Prob. of failure

$P(X = x) = {}^nC_x\, p^x\, q^{n-x}$

Learning CORNER

(iii) We have, $n = 90$, $p = 0.3$, $q = 0.7$

Mean, $\mu = np = 90 \times 0.3 = 27$

Varience, $\sigma^2 = npq = 90 \times 0.3 \times .7 = 18.9$

Let X be the function of normal distribution, i.e. $X \sim N(27,\ 18.9)$

$P(X > 35) + P(X < 27)$

applying continuity correction

$P(X > 35.5) + P(X < 26.5)$ $X \sim N(27,\ 18.9)$

$= P\left(\frac{X-\mu}{\sigma} > \frac{35.5-27}{\sqrt{18.9}}\right) + P\left(\frac{X-\mu}{\sigma} < \frac{26.5-27}{\sqrt{18.9}}\right)$ using $\frac{X-\mu}{\sigma} = Z$

$= P\left(Z > \frac{8.5}{\sqrt{18.9}}\right) + P\left(Z < \frac{-0.5}{\sqrt{18.9}}\right)$

$= 1 - \Phi\left(\frac{8.5}{\sqrt{18.9}}\right) + \Phi\left(\frac{-0.5}{\sqrt{18.9}}\right)$

$= 1 - \Phi(1.955) + 1 - \Phi(0.115)$

$= 2 - \Phi(1.955) - \Phi(0.115) = 0.48$ **(Ans).**

Note that:

Normal distribution may be used to approximate a Binomial distribution when n is large and $np > 5$, $nq > 5$.

Remember to apply continuity correction when approximating a Binomial distribution (discrete) by a Normal distribution (continuous).

Question 5

In Scotland, in November, on average 80% of days are cloudy. Assume that the weather on any one day is independent of the weather on other days.

(i) Use a normal approximation to find the probability of there being fewer than 25 cloudy days in Scotland in November (30 days). [4]

(ii) Give a reason why the use of a normal approximation is justified. [1]

[J11/P6/Q2]

Suggested Solution:

(i) $P(\text{a day is cloudy}) = p = 0.8$, $P(\text{a day is not cloudy}) = q = 0.2$

number of days in November, $n = 30$

X is a function of Binomial distribution, i.e. $X \sim B(30,\ 0.8)$

using a normal approximation,

$\mu = np = 30 \times 0.8 = 24$ and $\sigma^2 = npq = 30 \times 0.8 \times 0.2 = 4.8$

Let X be the function of normal distribution, i.e. $X \sim N(24,\ 4.8)$

for $P(X < 25)$, $X \sim N(24,\ 4.8)$

applying continuity correction

$P(X < 24.5)$

$\Rightarrow P\left(\frac{X-\mu}{\sigma} < \frac{24.5-24}{\sqrt{4.8}}\right)$ standardising X using $Z = \frac{X-\mu}{\sigma}$

$= P\left(Z < \frac{0.5}{\sqrt{4.8}}\right)$

$= P(Z < 0.228)$

$= \Phi(0.228) = 0.5902 \approx 0.590$ (3sf) **(Ans).**

(ii) $n = 30$, $p = 0.8$, $q = 0.2$

$np = 30 \times 0.8 = 24$, $nq = 30 \times 0.2 = 6$

since $np > 5$ and $nq > 5$,

$\therefore$ binomial model can be approximated as normal model.

Learning CORNER

Question 6

The lengths, in centimetres, of drinking straws produced in a factory have a normal distribution with mean μ and variance 0.64. It is given that 10% of the straws are shorter than 20 cm.

(i) Find the value of μ. [3]

(ii) Find the probability that, of 4 straws chosen at random, fewer than 2 will have a length between 21.5 cm and 22.5 cm. [6]

[J11/P6/Q6]

Suggested Solution:

(i) Let X be the length in centimeters of drinking straws which is a function of normal distribution i.e. $X \sim N(\mu, \sigma^2)$

$\therefore$ mean $= \mu$ and varience $\sigma^2 = 0.64$

$P(X < 20) = 10\%$

$\Rightarrow P\left(\frac{X-\mu}{\sigma} < \frac{20-\mu}{\sigma}\right) = 0.1$ standardizing X using $\frac{X-\mu}{\sigma} = Z$

$P\left(Z < \frac{20-\mu}{\sigma}\right) = 0.1$ $\because Z \sim N(0, 1)$

Let $\frac{20-\mu}{\sigma} = t$

$\Rightarrow P(Z < t) = 0.1$

$\Phi(-t) = 0.1$ $\because$ t is negative

$1 - \Phi(t) = 0.1$

$\Phi(t) = 0.9$

$\Phi(t) = \Phi(1.282)$

$\therefore t = -1.282$

$\Rightarrow \frac{20-\mu}{\sigma} = -1.282$

$\Rightarrow 20 - \mu = -1.282\sigma \Rightarrow \mu = 20 + (1.282)(0.8) = 21.025 \approx 21$ **(Ans)**

Candidates should know the parameters of the various distributions and what it represents.

For $X \sim N(\mu, \sigma^2)$, μ is the mean and σ is the standard deviation.

(ii) P(length between 21.5 cm and 22.5 cm)

$\Rightarrow P(21.5 < X < 22.5)$ $X \sim N(21.03, 0.64)$

$= P\left(\frac{21.5-21.03}{0.8} < \frac{X-\mu}{\sigma} < \frac{22.5-21.03}{0.8}\right)$

$= P(0.5875 < Z < 1.8375)$

$= \Phi(1.836) - \Phi(0.588)$

$= 0.9670 - 0.7217$

$= 0.2453$

Let X be the function of Binomial distribution, i.e. $X \sim B(4, 0.2453)$

$P(X < 2) = P(0) + P(1)$

$= {}^4C_0(0.2453)^0(0.7547)^4 + {}^4C_1(0.2453)^1(0.7547)^3$

$= 0.746$ **(Ans).**

Learning CORNER

Question 7

A triangular spinner has one red side, one blue side and one green side. The red side is weighted so that the spinner is four times more likely to land on the red side than on the blue side. The green side is weighted so that the spinner is three times more likely to land on the green side than on the blue side.

(i) Show that the probability that the spinner lands on the blue side is $\frac{1}{8}$. [1]

(ii) The spinner is spun 3 times. Find the probability that it lands on a different coloured side each time. [3]

(iii) The spinner is spun 136 times. Use a suitable approximation to find the probability that it lands on the blue side fewer than 20 times. [5]

[N11/P6/Q5]

Suggested Solution:

(i) Let the probability that spinner land on blue side $= a$

$\Rightarrow$ $P(\text{spinner land on red side}) = 4a$

$P(\text{spinner land on green side}) = 3a$

$P(\text{red}) + P(\text{green}) + P(\text{blue}) = 1$

$\Rightarrow \quad 4a + 3a + a = 1$

$8a = 1 \quad \Rightarrow \quad a = \dfrac{1}{8}$

hence, $P(\text{spinner land on blue side}) = \dfrac{1}{8}$ **(Ans).**

(ii) $P(\text{spinner land on different coloured side each time})$

$= {}^3P_3\big(P(\text{red}) \times P(\text{green}) \times P(\text{blue})\big)$

$= 6\left(\dfrac{1}{8}\right)\left(\dfrac{4}{8}\right)\left(\dfrac{3}{8}\right) = \dfrac{9}{64}$ **(Ans).**

(iii) No. of trials, $n = 136$

$P(\text{spinner lands on blue side}) = p = \dfrac{1}{8}$

$\Rightarrow \quad q = \dfrac{7}{8}$

Let X be the number of times that spinner lands on blue side.

$\therefore$ X is a function of binomial distribution i.e. $X \sim B\left(136, \frac{1}{8}\right)$

applying normal approximation to binomial distribution,

$\mu = np = 136 \times \frac{1}{8} = 17$

$\sigma^2 = npq = 136 \times \frac{1}{8} \times \frac{7}{8} = \frac{119}{8}$

for $P(X < 20)$

applying continuity correction,

$P(X < 19.5)$, $\qquad X \sim N\left(17, \frac{119}{8}\right)$

$\Rightarrow \quad P\left(\dfrac{X - \mu}{\sigma} < \dfrac{19.5 - 17}{\sqrt{\frac{119}{8}}}\right)$ standardising X to Z

$= \quad P(Z < 0.648) \qquad \because \ Z \sim N(0, 1)$

$= \quad \Phi(0.648) = 0.7415 \approx 0.742$ **(Ans).**

Note that:

$X \sim \text{Bin}\ (n, p)$

$n =$ Total no. of trials

$p =$ Prob. of success

$q =$ Prob. of failure

$P(X = x) = {}^nC_x\, p^x\, q^{n-x}$

Note that:

Normal distribution may be used to approximate a Binomial distribution when n is large and $np > 5$, $nq > 5$.

Remember to apply continuity correction when approximating a Binomial distribution (discrete) by a Normal distribution (continuous).

Learning CORNER

Question 34

The heights, in metres, of fir trees in a large forest have a normal distribution with mean 40 and standard deviation 8.

(i) Find the probability that a fir tree chosen at random in this forest has a height less than 45 metres. [2]

(ii) Find the probability that a fir tree chosen at random in this forest has a height within 5 metres of the mean. [2]

In another forest, the heights of another type of fir tree are modelled by a normal distribution. A scientist measures the heights of 500 randomly chosen trees of this type. He finds that 48 trees are less than 10m high and 76 trees are more than 24m high.

(iii) Find the mean and standard deviation of the heights of trees of this type. [5]

[N19/P6/Q6]

Suggested Solution:

(i) Let X height of fir tree, be a function of Normal Distribution, $X \sim N(40,\ 8^2)$

$P(X < 45)$

$\Rightarrow \ P\left(\frac{X-\mu}{\sigma} < \frac{45-40}{8}\right)$ Standardizing X, using $Z = \frac{X-\mu}{\sigma}$

$\Rightarrow \ P(Z < 0.625)$ $Z \sim N(0,\ 1)$

$= \Phi(0.625)$ $P(Z < a) = \Phi(a)$

$= 0.734$ **(Ans).**

Candidates should know the parameters of the various distributions and what it represents. For $X \sim N(\mu,\ \sigma^2)$, μ is the mean and σ is the standard deviation.

(ii) $P((40-5) < X < (40+5))$ $X \sim N(40,\ 8^2)$

$\Rightarrow \ P(35 < X < 45)$

$\Rightarrow \ P\left(\frac{35-40}{8} < \frac{X-\mu}{\sigma} < \frac{45-40}{8}\right)$ Standardizing X, using $Z = \frac{X-\mu}{\sigma}$

$\Rightarrow \ P(-0.625 < Z < 0.625)$ $Z \sim N(0,\ 1)$

$= \Phi(0.625) - \Phi(-0.625)$ $P(a < Z < b) = \Phi(b) - \Phi(a)$

$= \Phi(0.625) - [1 - \Phi(0.625)]$ $\Phi(-a) = 1 - \Phi(a)$

$= 2\Phi(0.625) - 1$

$= 2(0.7340) - 1$

$= 0.468$ **(Ans).**

(iii) Let μ be the mean and σ be the standard deviation of the distribution

$P(X < 10) = \frac{48}{500}$ $\quad$ $X \sim N(\mu,\ \sigma^2)$

$\Rightarrow \quad P\left(\frac{X-\mu}{\sigma} < \frac{10-\mu}{\sigma}\right) = 0.096$ $\quad$ Standardizing X, using $\frac{X-\mu}{\sigma} = Z$

$\Rightarrow \quad P\left(Z < \frac{10-\mu}{\sigma}\right) = 0.096$ $\quad$ $Z \sim N(0,\ 1)$

Let $a = \frac{10-\mu}{\sigma}$

$\Rightarrow \quad P(Z < a) = 0.096$

$\because \quad a$ is negative

$\Rightarrow \quad P(Z < -a) = 0.096$

$\Rightarrow \quad \Phi(-a) = 0.096$ $\quad$ $P(Z < a) = \Phi(a)$

$\Rightarrow \quad 1 - \Phi(a) = 0.096$ $\quad$ $\Phi(-a) = 1 - \Phi(a)$

$\Rightarrow \quad \Phi(a) = 0.9040 \quad \Rightarrow \quad a = -1.305$

$\therefore \quad \frac{10-\mu}{\sigma} = -1.305 \quad \Rightarrow \quad 10 - \mu = -1.305\sigma \ \cdots\cdots\ (1)$

$\Rightarrow \quad P(X > 24) = \frac{76}{500}$

$\Rightarrow \quad P\left(\frac{X-\mu}{\sigma} > \frac{24-\mu}{\sigma}\right) = 0.152$ $\quad$ Standardizing X, using $\frac{X-\mu}{\sigma} = Z$

$\Rightarrow \quad P\left(Z > \frac{24-\mu}{\sigma}\right) = 0.152$ $\quad$ $Z \sim N(0,\ 1)$

Let $b = \frac{24-\mu}{\sigma}$

$\Rightarrow \quad P(Z > b) = 0.152$

$\Rightarrow \quad 1 - \Phi(b) = 0.152$

$\Rightarrow \quad \Phi(b) = 0.8480 \quad \Rightarrow \quad b = 1.028$

$\therefore \quad \frac{24-\mu}{\sigma} = 1.028 \quad \Rightarrow \quad 24 - \mu = 1.028\sigma \ \cdots\cdots\ (2)$

Substitute equation (1) into equation (2) and solving simultaneously

$24 - (10 + 1.305\sigma) = 1.028\sigma$

$14 = 2.333\sigma \quad \Rightarrow \quad \sigma = 6$ **(Ans).**

$\mu = 10 + 1.305(6) = 17.8$ **(Ans).**

Question 35

Trees in the Redian forest are classified as tall, mediumor short, according to their height. The heights can be modelled by a normal distribution with mean 40 m and standard deviation 12 m. Trees with a height of less than 25 m are classified as short.

(a) Find the probability that a randomly chosen tree is classified as short. [3]

Of the trees that are classified as tall or medium, one third are tall and two thirds are medium.

(b) Show that the probability that a randomly chosen tree is classified as tall is 0.298, correct to 3 decimal places. [2]

(c) Find the height above which trees are classified as tall. [3]

[J20/P5/Q4]

Learning CORNER

Suggested Solution:

(a) Mean, $\mu = 40$ m, Standard deviation, $\sigma = 12$ m

$P(\text{classified as short}) = P(X < 25)$

$\Rightarrow P\left(\dfrac{X-\mu}{\sigma} < \dfrac{25-40}{12}\right)$ Standardizing X, using $\dfrac{X-\mu}{\sigma} = Z$

$\Rightarrow P(Z < -1.25)$ $\because Z \sim N(0, 1)$

$= \Phi(-1.25)$ $P(Z < a) = \Phi(a)$

$= 1 - \Phi(1.25)$ $\Phi(-a) = 1 - \Phi(a)$

$= 1 - 0.8944$

$= 0.1056 \approx 0.106$ **(Ans).**

Candidates should know the parameters of the various distributions and what it represents.

For $X \sim N(\mu, \sigma^2)$,
μ is the mean and
σ is the standard deviation.

(b) $P(\text{tree classified as tall or medium}) = 1 - 0.1056 = 0.8944$

$\therefore P(\text{tree classified as tall}) = \dfrac{1}{3} \times 0.8944 = 0.298$ **(Ans).**

(c) Let h be the height above which trees are classified as tall.

$P(X > h) = 0.298$

$\Rightarrow P\left(\dfrac{X-\mu}{\sigma} > \dfrac{h-40}{12}\right) = 0.298$

Let $\dfrac{h-40}{12} = t$

$\Rightarrow P(Z > t) = 0.298$

$\Rightarrow 1 - \Phi(t) = 0.2980$

$\Rightarrow \Phi(t) = 0.7020$

$\Rightarrow t = \Phi^{-1}(0.7020) = 0.531$

$\therefore \dfrac{h-40}{12} = 0.531 \Rightarrow h = 46.4$ cm **(Ans).**

Question 36

On any given day, the probability that Moena messages her friend Pasha is 0.72.

(a) Find the probability that for a random sample of 12 days Moena messages Pasha on no more than 9 days. [3]

(b) Moena messages Pasha on 1 January. Find the probability that the next day on which she messages Pasha is 5 January. [1]

(c) Use an approximation to find the probability that in any period of 100 days Moena messages Pasha on fewer than 64 days. [5]

[J20/P5/Q7]

Learning CORNER

Suggested Solution:

(a) Number of days, $n = 12$

$P(\text{Moena messages}) = p = 0.72$

$P(\text{Moena does not message}) = q = 0.28$

Let X no. of days be the function of Binomial Distribution

$P(X \leq 9)$

$= 1 - [P(X = 10) + P(X = 11) + P(X = 12)]$

$= 1 - \left[{}^{12}C_{10}(0.72)^{10}(0.28)^2 + {}^{12}C_{11}(0.72)^{11}(0.28)^1 + {}^{12}C_{12}(0.72)^{12}(0.28)^0 \right]$

$= 1 - 0.3037 = 0.69629 \approx 0.696$ (3 sf) **(Ans).**

(b) $P(\text{next message is on 5 January}) = (0.28)(0.28)(0.28)(0.72) = 0.0158$ **(Ans).**

(c) Total number of days, $n = 100$

$P(\text{Moena messages}) = p = 0.72$

$P(\text{Moena does not message}) = q = 0.28$

$X \sim B(100,\ 0.72)$

Binomial Distribution can be approximated as Normal Distribution

$\mu = np = 100 \times 0.72 = 72$

$\sigma^2 = npq = 100 \times 0.72 \times 0.28 = 20.16$

$X \sim N(72,\ 20.16)$

For $P(X < 64)$

Applying continuity correction

$\Rightarrow\ P(X < 63.5)$ $\qquad X \sim N(72,\ 20.16)$

$\Rightarrow\ P\left(\dfrac{X - \mu}{\sigma} < \dfrac{63.5 - 72}{\sqrt{20.16}} \right)$ Standardizing X, using $\dfrac{X - \mu}{\sigma} = Z$

$\Rightarrow\ P(Z < -1.893)$ $\qquad \because\ Z \sim N(0,\ 1)$

$= \Phi(-1.893)$

$= 1 - \Phi(1.893) = 1 - (0.9708) = 0.0292$ **(Ans).**

Note that:

Normal distribution may be used to approximate a Binomial distribution when n is large and $np > 5$, $nq > 5$.

Remember to apply continuity correction when approximating a Binomial distribution (discrete) by a Normal distribution (continuous).

Question 37

Pia runs 2 km every day and her times in minutes are normally distributed with mean 10.1 and standard deviation 1.3.

(a) Find the probability that on a randomly chosen day Pia takes longer than 11.3 minutes to run 2 km. [3]

(b) On 75% of days, Pia takes longer than t minutes to run 2 km. Find the value of t. [3]

(c) On how many days in a period of 90 days would you expect Pia to take between 8.9 and 11.3 minutes to run 2 km? [3]

[N20/P5/Q3]

Suggested Solution:

(a) Let X be the time to run 2 km follow normal distribution with

Mean, $\mu = 10.1$, Standard deviation, $\sigma = 1.3$

i.e, $X \sim N(10.1,\ (1.3)^2)$

P(Pia takes longer than 11.3 minutes to run 2 km)

$= P(X > 11.3)$

$\Rightarrow P\left(X - \mu > \dfrac{11.3 - 10.1}{1.3}\right)$ Standardizing X, using $\dfrac{X-\mu}{\sigma} = Z$

$\Rightarrow P(Z > 0.923)$ Using $P(Z > a) = 1 - \Phi(a)$

$= 1 - \Phi(0.923)$

$= 1 - 0.8220 = 0.1780$ **(Ans).**

(b) $P(X > t) = 75\%$ $X \sim N(10.1,\ 1.3^2)$

$\Rightarrow P\left(\dfrac{X-\mu}{\sigma} > \dfrac{t - 10.1}{1.3}\right) = 0.75$ Standardizing X, using $\dfrac{X-\mu}{\sigma} = Z$

Let $\dfrac{t - 10.1}{1.3} = a$

$\Rightarrow P(Z > a) = 0.75$

$\Rightarrow P(Z > -a) = 0.75$ since a is negative

$\Rightarrow 1 - \Phi(-a) = 0.75$ Using $P(Z > a) = 1 - \Phi(a)$

$\Rightarrow 1 - [1 - \Phi(a)] = 0.75$ Using $\Phi(-a) = 1 - \Phi(a)$

$\Rightarrow \Phi(a) = 0.75$

$\Rightarrow a = -0.674$

$\therefore \dfrac{t - 10.1}{1.3} = -0.674 \Rightarrow t = 9.22$ minutes **(Ans).**

(c) Mean, $\mu = 10.1$, Standard deviation, $\sigma = 1.3$

$P(8.9 < X < 11.3)$

$\Rightarrow P\left(\dfrac{8.9 - 10.1}{1.3} < \dfrac{X-\mu}{\sigma} < \dfrac{11.3 - 10.1}{1.3}\right)$ Standardizing X using $\dfrac{X-\mu}{\sigma} = Z$

$\Rightarrow P(-0.923 < Z < 0.923)$

$= 2\Phi(0.923) - 1$ Using $P(-a < Z < a) = 2\Phi(a) - 1$

$= 2(0.8220) - 1 = 0.644$

Expected number of days $= E(X) = np$

$= 90 \times 0.644$

$= 57.96 \approx 57$ days **(Ans).**

Question 38

The weights of bags of sugar are normally distributed with mean 1.04 kg and standard deviation σ kg. In a random sample of 2000 bags of sugar, 72 weighed more than 1.10 kg.

Find the value of σ. [4]

[J21/P5/Q2]

Suggested Solution:

Let X be the function of normal distribution i.e, $X \sim N(1.04,\ \sigma^2)$

$P(X > 1.10) = \dfrac{72}{2000}$

$\Rightarrow\ P\left(\dfrac{X-\mu}{\sigma} > \dfrac{1.10-1.04}{\sigma}\right) = 0.036$ Standardizing X, using $Z = \dfrac{X-\mu}{\sigma}$

$\Rightarrow\ P\left(Z > \dfrac{0.06}{\sigma}\right) = 0.036$

Let $\dfrac{0.06}{\sigma} = t$

$\Rightarrow\ P(Z > t) = 0.036$

since t is $+\text{ve}$

$\Rightarrow\ 1 - \Phi(t) = 0.036$

$\Rightarrow\ \Phi(t) = 0.964 \quad \Rightarrow \quad t = \Phi^{-1}(0.9640) = 1.798$

$\therefore\ \dfrac{0.06}{\sigma} = 1.798 \quad \Rightarrow \quad \sigma = 0.03337 \approx 0.0334$ **(Ans).**

Question 39

Every day Richard takes a flight between Astan and Bejin. On any day, the probability that the flight arrives early is 0.15, the probability that it arrives on time is 0.55 and the probability that it arrives late is 0.3.

(a) Find the probability that on each of 3 randomly chosen days, Richard's flight does not arrive late. [1]

(b) Find the probability that for 9 randomly chosen days, Richard's flight arrives early at least 3 times. [3]

(c) 60 days are chosen at random.

Use an approximation to find the probability that Richard's flight arrives early at least 12 times. [5]

[J21/P5/Q5]

Suggested Solution:

(a) Total number of days, $n = 3$

$P(\text{arrive late}) = p = 0.3$

$P(\text{does not arrive late}) = q = 0.7$

Let X be a function of Binomial distribution i.e. $X \sim B$ in $(3,\ 0.3)$

$P(X = 0) = {}^{3}C_0(0.3)^0(0.7)^3 = 0.343$ **(Ans).**

(b) Total number of days, $n = 9$

$P(\text{flight arrives early}) = p = 0.15$

$P(\text{flight does not arrive early}) = q = 0.85$

Let X be a function of Binomial distribution i.e. $X \sim B$ in $(9,\ 0.15)$

$$P(X \geq 3) = 1 - [P(X=0) + P(X=1) + P(X=2)]$$

$$= 1 - \left[{}^{9}C_0(0.15)^0(0.85)^9 + {}^{9}C_1(0.15)^1(0.85)^8 + {}^{9}C_2(0.15)^2(0.85)^7\right]$$

$= 0.1408 \approx 0.141$ **(Ans).**

(c) Total number of days, $n = 60$

P(flight arrives early) $= p = 0.15$

P(flight does not arrive early) $= q = 0.85$

$X \sim B$ in $(60,\ 0.15)$

X is the functionof Normal Distribution, i.e $X \sim N(\mu,\ \sigma^2)$

$\mu = np = 60 \times 0.15 = 9$

$\sigma^2 = npq = (60)(0.15)(0.85) = 7.65$

$\Rightarrow\ X \sim N(9,\ 7.65)$

For $P(X \geq 12)$

Applying continuity correction

$\Rightarrow\ P(X > 11.5)$ $\qquad X \sim N(9,\ 7.65)$

$\Rightarrow\ P\left(\dfrac{X-\mu}{\sigma} > \dfrac{11.5-9}{\sqrt{7.65}}\right)$ $\qquad$ Standardizing X, using $\dfrac{X-\mu}{\sigma} = Z$

$\Rightarrow\ P(Z > 0.904)$ $\qquad \because\ Z \sim N(0,\ 1)$

$= 1 - \Phi(0.904)$

$= 1 - 0.8169 = 0.1831 \approx 0.183$ **(Ans).**

Question 40

The times taken, in minutes, to complete a particular task by employees at a large company are normally distributed with mean 32.2 and standard deviation 9.6.

(a) Find the probability that a randomly chosen employee takes more than 28.6 minutes to complete the task. [3]

(b) 20% of employees take longer than t minutes to complete the task.

Find the value of t. [3]

(c) Find the probability that the time taken to complete the task by a randomly chosen employee differs from the mean by less than 15.0 minutes. [4]

[N21/P5/Q6]

Suggested Solution:

(a) Let T be the function of normal distribution i.e, $T \sim N(32.2,\ 9.6^2)$

$P(T > 28.6)$

$\Rightarrow\ P\left(\dfrac{T-\mu}{\sigma} > \dfrac{28.6-32.2}{9.6}\right)$ $\qquad$ Standardizing T, using $\dfrac{T-\mu}{\sigma} = Z$

$\Rightarrow\ P(Z > -0.375)$ $\qquad Z \sim N(0,\ 1)$

$= 1 - \Phi(-0.375)$ $\qquad P(Z > a) = 1 - \Phi(a)$

$= 1 - [1 - \Phi(0.375)] = \Phi(0.375)$ $\qquad \Phi(-a) = 1 - \Phi(a)$

$= 0.6462$ $\qquad$ Using noraml distribution table

$= 0.646$ **(Ans).**

(b) $P(T > t) = 0.2$ $\qquad T \sim N(32.2,\ 9.6^2)$

$\Rightarrow\ P\left(\dfrac{T-\mu}{\sigma} > \dfrac{t-32.2}{9.6}\right) = 0.2$ $\qquad$ Standardizing T, using $\dfrac{T-\mu}{\sigma} = Z$

Let $a = \dfrac{t-32.2}{9.6}$ where a is +ve

$\Rightarrow\ P(Z > a) = 0.2$ $\qquad Z \sim N(0,\ 1)$

$\Rightarrow\ 1-\Phi(a) = 0.2$ $\qquad$ Using $P(Z > a) = 1-\Phi(a)$

$\Rightarrow\ \Phi(a) = 0.8$

$\Rightarrow\ \Phi(a) = \Phi(0.842)$ $\qquad$ Using Normal disbribution table

$\Rightarrow\ a = 0.842$

$\therefore\ \dfrac{t-32.2}{9.6} = 0.842$

$\Rightarrow\ t = 40.28 \approx 40.3$ minutes **(Ans).**

(c) $P(\mu-15 < T < \mu+15)$ $\qquad T \sim N(32.2,\ 9.6)$

$\Rightarrow\ P\left(\dfrac{\mu-15-32.2}{\sigma} < \dfrac{T-\mu}{\sigma} < \dfrac{\mu+15-32.2}{9.6}\right)$

$\Rightarrow\ P\left(\dfrac{-15}{9.6} < Z < \dfrac{15}{9.6}\right)$ $\qquad Z \sim N(0,\ 1)$

$= \Phi\left(\dfrac{15}{9.6}\right) - \Phi\left(\dfrac{-15}{9.6}\right)$ $\qquad P(a < Z < b) = \Phi(b) - \Phi(a)$

$= \Phi\left(\dfrac{15}{9.6}\right) - \left[1-\Phi\left(\dfrac{15}{9.6}\right)\right]$ $\qquad \Phi(-a) = 1-\Phi(a)$

$= 2\Phi\left(\dfrac{15}{9.6}\right) - 1$

$= 2\Phi(1.563) - 1$

$= 2(0.9410) - 1 = 0.882$ **(Ans).**

Question 41

The weights, in kg, of bags of rice produced by Anders have the distribution $N(2.02, 0.03^2)$.

(a) Find the probability that a randomly chosen bag of rice produced by Anders weighs between 1.98 and 2.03 kg. [3]

The weights of bags of rice produced by Binders are normally distributed with mean 2.55 kg and standard deviation σ kg. In a random sample of 5000 of these bags, 134 weighed more than 2.6 kg.

(b) Find the value of σ. [4]

[J22/P5/Q4]

Suggested Solution:

(a) Let W kg be the weight of bags of rice produced by Anders,

which is a function of normal distribution i.e, $W \sim N(2.02,\ 0.03^2)$

$P(1.98 < W < 2.03)$

$$\Rightarrow\ P\left(\frac{1.98-2.02}{0.03} < \frac{W-\mu}{\sigma} < \frac{2.03-2.02}{0.03}\right)$$

$\Rightarrow\ P(-1.333 < Z < 0.333)$ $\qquad Z \sim N(0,\ 1)$

$= \Phi(0.333) - \Phi(-1.333)$ $\qquad P(a < Z < b) = \Phi(b) - \Phi(a)$

$= \Phi(0.333) - [1 - \Phi(1.333)]$ $\qquad \Phi(-b) = 1 - \Phi(b)$

$= \Phi(0.333) - 1 + \Phi(1.333)$

$= 0.9087 - 1 + 0.6304 = 0.5391 \approx 0.539$ **(Ans).**

(b) Let X kg be the weight of bags of rice produced by Binders

which is a function of normal distribution i.e, $X \sim N(2.55,\ \sigma^2)$

$P(X \text{ is more than } 2.6) = \dfrac{134}{5000}$

$\Rightarrow\ P(X > 2.6) = \dfrac{134}{5000}$

$$\Rightarrow\ P\left(\frac{X-\mu}{\sigma} > \frac{2.6-2.55}{\sigma}\right) = \frac{134}{5000}$$

$\Rightarrow\ P\left(Z > \dfrac{0.05}{\sigma}\right) = 0.0268$ $\qquad Z \sim N(0,\ 1)$

Let $\dfrac{0.05}{\sigma} = t$

$\Rightarrow\ P(Z > t) = 0.0268$ $\qquad (t \text{ is } +\text{ve})$

$\Rightarrow\ 1 - \Phi(t) = 0.0268$ $\qquad P(Z > a) = 1 - \Phi(a)$

$\Rightarrow\ \Phi(t) = 0.9732 \ \Rightarrow\ t = \Phi^{-1}(0.9732) \ \Rightarrow\ t = 1.93$

$\therefore\ \dfrac{0.05}{\sigma} = 1.93 \ \Rightarrow\ \sigma = 0.025907 \approx 0.0259$ **(Ans).**

Candidates should know the parameters of the various distributions and what it represents.

For $X \sim N(\mu,\ \sigma^2)$, μ is the mean and σ is the standard deviation.

Question 42

In a large college, 28% of the students do not play any musical instrument, 52% play exactly one musical instrument and the remainder play two or more musical instruments.

A random sample of 12 students from the college is chosen.

(a) Find the probability that more than 9 of these students play at least one musical instrument. [3]

A random sample of 90 students from the college is now chosen.

(b) Use an approximation to find the probability that fewer than 40 of these students play exactly one musical instrument. [5]

[J22/P5/Q5]

Suggested Solution:

(a) Let X be the number of students who play at least one musical instrument which is a function of binomial distribution.

Total number of students, $n = 12$

Probability that a student play at least one musical instrument, $p = 0.72$

Probability that a student does not play any musical instrument, $q = 0.28$

i.e. $X \sim B(12,\ 0.72)$

P(more than 9 students play at least one musical instrument)

$= P(X = 10) + P(X = 11) + P(X = 12)$

$= {}^{12}C_{10}(0.72)^{10}(0.28)^2 + {}^{12}C_{11}(0.72)^{11}(0.28) + {}^{12}C_{12}(0.72)^{12}$

$= 0.193725 + 0.090573 + 0.019408$

$= 0.303706 \approx 0.304$ **(Ans).**

Note that:

Normal distribution may be used to approximate a Binomial distribution when n is large and $np > 5$, $nq > 5$.

(b) Total number of students, $n = 90$

P(student plays exactly one musical instrument) $= p = 0.52$

P(student does not play exactly one musical instrument) $= q = 0.48$

$X \sim B\ (90,\ 0.52)$

$np = (90)(0.52) = 46.8, \quad nq = (90)(0.48) = 43.2$

$\because\ np > 5 \quad \& \quad nq > 5$

Binomial distribution can be approximated as normal distribution,

Remember to apply continuity correction when approximating a Binomial distribution (discrete) by a Normal distribution (continuous).

i.e $X \sim N(\mu,\ \sigma^2)$

$\mu = np = 46.8, \qquad \sigma^2 = npq = (90)(0.52)(0.48) = 22.46$

$\Rightarrow\ X \sim N(46.8,\ 22.46)$

For $P(X < 40)$

Applying continuity correction

$P(X < 39.5)$	$X \sim N(46.8,\ 22.46)$
$\Rightarrow\ P\left(\dfrac{X-\mu}{\sigma} < \dfrac{39.5 - 46.8}{\sqrt{22.46}}\right)$	Standardizing X, using $\dfrac{X-\mu}{\sigma} = Z$
$\Rightarrow\ P(Z < -1.540)$	$Z \sim N(0,\ 1)$
$= \Phi(-1.540)$	$P(Z < a) = \Phi(a)$

$= 1 - \Phi(1.540)$

$= 1 - (0.9382) = 0.0618$ **(Ans).**

Question 43

The lengths of the rods produced by a company are normally distributed with mean 55.6 mm and standard deviation 1.2 mm.

(a) In a random sample of 400 of these rods, how many would you expect to have length less than 54.8 mm? [4]

(b) Find the probability that a randomly chosen rod produced by this company has a length that is within half a standard deviation of the mean. [3]

[N22/P5/Q2]

Suggested Solution:

(a) Mean, $\mu = 55.6$ mm Standard deviation, $\sigma = 1.2$ mm

Let the number of rods X be a function of normal distribution

P(Length is less than 54.8 mm)

$P(X < 54.8)$ $X \sim N(55.6,\ 1.2^2)$

$= P\left(\dfrac{X-\mu}{\sigma} < \dfrac{54.8-55.6}{1.2}\right)$ Standardizing X, using $Z = \dfrac{X-\mu}{\sigma}$

$= P(Z < -0.667)$ $Z \sim N(0,\ 1)$

$= \Phi(-0.667)$ $\Phi(Z < a) = \Phi(a)$

$= 1 - \Phi(0.667)$ $\Phi(-a) = 1 - \Phi(a)$

$= 1 - 0.7477 = 0.2523$

Expected number of rods $= (0.2523) \times 400 = 100.92 \approx 101$ **(Ans).**

Candidates should know the parameters of the various distributions and what it represents.

For $X \sim N(\mu,\ \sigma^2)$, μ is the mean and σ is the standard deviation.

(b) $P\left(\mu - \frac{1}{2}\sigma < X < \mu + \frac{1}{2}\sigma\right)$ $X \sim N(55.6,\ 1.2^2)$

$= P\left(\dfrac{\mu - \frac{1}{2}\sigma - \mu}{\sigma} < \dfrac{X-\mu}{\sigma} < \dfrac{\mu + \frac{1}{2}\sigma - \mu}{\sigma}\right)$

$= P(-0.5 < Z < 0.5)$ $Z \sim N(0,\ 1)$

$= \Phi(0.5 - \Phi(-0.5)$ $P(a < Z < b) = \Phi(b) - \Phi(a)$

$= \Phi(0.5) - [1 - \Phi(0.5)]$ $\Phi(-b) = 1 - \Phi(b)$

$= 2\Phi(0.5) - 1 = 2(0.6915) - 1 = 0.383$ **(Ans).**

Question 44

At a company's call centre, 90% of callers are connected immediately to a representative.

A random sample of 12 callers is chosen.

(a) Find the probability that fewer than 10 of these callers are connected immediately. [3]

A random sample of 80 callers is chosen.

(b) Use an approximation to find the probability that more than 69 of these callers are connected immediately. [5]

(c) Justify the use of your approximation in part **(b)**. [1]

[N22/P5/Q6]

Suggested Solution:

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(a) Let X be the function of binomial distribution

P(callers are connected immediately) $= p = 0.9$

P(callers are not connected immediately) $= q$

$= 1 - p = 0.1$

Number of callers $= n = 12$

P(fewer than 10 callers connected immediately)

$= P(X < 10)$

$= P(X = 0, 1, 2, 3, \ldots\ldots, 9)$

$= 1 - [P(X = 10) \text{ or } (X = 11) \text{ or } (X = 12)]$

$= 1 - \left[{}^{12}C_{10}(0.9)^{10}(0.1)^2 + {}^{12}C_{11}(0.9)^{11}(0.1)^1 + {}^{12}C_{12}(0.9)^{12}(0.1)^0 \right]$

$= 1 - 0.8891 = 0.1108 \approx 0.111$ **(Ans).**

Note that:

$X \sim \text{Bin}\ (n, p)$

n = Total no. of trials

p = Prob. of success

q = Prob. of failure

$P(X = x) = {}^nC_x\, p^x\, q^{n-x}$

(b) Let X be the function of binomial distribution

$n = 80$

P(callers are connected immediately) $= p = 0.9$

P(callers are not connected immediately) $= q$

$= 1 - 0.9 = 0.1$

$\Rightarrow \quad X \sim B\ (80, 0.9)$

P(more than 69 callers are connected immediately)

$P(X > 69)$ $\qquad X \sim B(80, 0.9)$

Applying normal approximation to the binomial distribution

Mean, $\mu = np$

$= 80 \times 0.9 = 72,$

Variance, $\sigma^2 = npq$

$= 80 \times 0.9 \times 0.1 = 7.2$

Applying continuity correction

$P(X > 69.5)$ $\qquad X \sim N(72, 7.2)$

$\Rightarrow \quad P\left(\dfrac{X - \mu}{\sigma} > \dfrac{69.5 - 72}{\sqrt{7.2}} \right)$ $\qquad$ Standardizing X, using $\dfrac{X - \mu}{\sigma} = Z$

$\Rightarrow \quad P(Z > -0.932)$ $\qquad Z \sim N(0, 1)$

$= 1 - \Phi(-0.932)$

$= 1 - [1 - \Phi(0.932)]$

$= \Phi(0.932) = 0.8243 \approx 0.824$ **(Ans).**

Note that:

Normal Distribution may be used to approximate a Binomial Distribution when n is large $np > 5$ and $nq > 5$.

(c) Since, $np = 80 \times 0.9 = 72$, $nq = 80 \times 0.1 = 8$

$\Rightarrow \quad np > 5$ and $nq > 5$

$\therefore$ Binomial distribution can be approximated to normal distribution

Question 47

(a) The heights of the members of a club are normally distributed with mean 166 cm and standard deviation 10 cm.

(i) Find the probability that a randomly chosen member of the club has height less than 170 cm. [2]

(ii) Given that 40% of the members have heights greater than h cm, find the value of h correct to 2 decimal places. [3]

(b) The random variable X is normally distributed with mean μ and standard deviation σ.

Given that $\sigma = \frac{2}{3}\mu$, find the probability that a randomly chosen value of X is positive. [3]

[N23/P5/Q5]

Suggested Solution:

(a) (i) Mean, $\mu = 166$, Standard deviation, $\sigma = 10$

Let X, be the heights of the members, is a function of normal distribution

i.e. $X \sim N(166,\ 10^2)$

$P(X < 170)$

$= P\left(\frac{X-\mu}{\sigma} < \frac{170-166}{10}\right)$ Change X to Z, using $Z = \frac{X-\mu}{\sigma}$

$= P(Z < 0.4)$ $Z \sim N(0,\ 1)$

$= \Phi(0.4)$ $P(Z < a) = \Phi(a)$

$= 0.6554 \approx 0.655$ (3 s.f) **(Ans).**

(ii) $P(X > h) = 40\%$ $X \sim N(166,\ 10^2)$

$\Rightarrow P\left(\frac{X-\mu}{\sigma} > \frac{h-166}{10}\right) = 0.4$ Change X to Z, using $Z = \frac{X-\mu}{\sigma}$

$\Rightarrow P\left(Z > \frac{h-166}{10}\right) = 0.4$ $Z \sim N(0,\ 1)$

Let $\frac{h-166}{10} = t$

$\Rightarrow P(Z > t) = 0.4$ t is $+$ve

$\Rightarrow 1 - \Phi(t) = 0.4$ $P(Z > a) = 1 - \Phi(a)$

$\Rightarrow \Phi(t) = 0.6$

$\Rightarrow t = 0.253$

$\Rightarrow \frac{h-166}{10} = 0.253$

$\Rightarrow h = 168.53$ **(Ans).**

(b) Mean = μ, Standard deviation = σ

Let X random variable, be a function of normal distribution

$P(X > 0)$ $\qquad X \sim N(\mu, \sigma^2)$

$\Rightarrow P\left(\dfrac{X-\mu}{\sigma} > \dfrac{-\mu}{\sigma}\right)$

$\Rightarrow P\left(Z > \dfrac{-\mu}{\frac{2}{3}\mu}\right)$ $\qquad \sigma = \dfrac{2}{3}\mu$ (given)

$= P(Z > -1.5)$ $\qquad Z \sim N(0, 1)$

$= 1 - \Phi(-1.5)$ $\qquad P(Z > a) = 1 - \Phi(a)$

$= 1 - [1 - \Phi(1.5)]$ $\qquad \Phi(-a) = 1 - \Phi(a)$

$= \Phi(1.5)$

$= 0.9332 \approx 0.933$ (3 s.f) **(Ans).**

Question 48

The weights of oranges can be modelled by a normal distribution with mean 131 grams and standard deviation 54 grams. Oranges are classified as small, medium or large. A large orange weighs at least 184 grams and 20% of oranges are classified as small.

(a) Find the percentage of oranges that are classified as large. [3]

(b) Find the greatest possible weight of a small orange. [3]

[J24/P5/Q3]

Suggested Solution:

(a) Let X be the weight of oranges is a function of normal distribution

Mean, $\mu = 131$, Standard deviation, $\sigma = 54$

P(oranges that are classified as large)

$\Rightarrow P(X > 184)$ $\qquad X \sim N(131, 54^2)$

$\Rightarrow P\left(\dfrac{X-\mu}{\sigma} > \dfrac{184-131}{54}\right)$ $\qquad$ Standardizing X using $Z = \dfrac{X-\mu}{\sigma}$

$\Rightarrow P(Z > 0.9815)$ $\qquad Z \sim N(0, 1)$

$\Rightarrow 1 - \Phi(0.9815)$ $\qquad P(Z > a) = 1 - \Phi(a)$

$\Rightarrow 1 - (0.8370) = 0.163$

$\therefore$ Percentage of oranges classified as large $= 0.163 \times 100\%$

$= 16.3\%$ **(Ans).**

(b) Let a be the greatest possible weight of a small orange

$P(X < a) = 0.20$ $\quad$ $X \sim N(131,\ 54^2)$

$\Rightarrow \quad P\left(\dfrac{X-\mu}{\sigma} < \dfrac{a-131}{54}\right) = 0.20$

$\Rightarrow \quad P\left(Z < \dfrac{a-131}{54}\right) = 0.20$ $\quad$ $Z \sim N(0,\ 1)$

Let, $\dfrac{a-131}{54} = t$

$\Rightarrow \quad P(Z < t) = 0.20$ $\quad$ (as t is -ve)

$\Rightarrow \quad P(Z < -t) = 0.20$

$\Rightarrow \quad \Phi(-t) = 0.20$ $\quad$ $P(Z < -t) = \Phi(-t)$

$\Rightarrow \quad 1 - \Phi(t) = 0.20$ $\quad$ $\Phi(-t) = 1 - \Phi t$

$\Rightarrow \quad \Phi(t) = 0.80$

$\Rightarrow \quad t = \Phi^{-1}(0.80)$

$\Rightarrow \quad t = 0.842$

$\therefore \quad t = -0.842$ $\quad$ (as t is -ve)

$\Rightarrow \quad \dfrac{a-131}{54} = -0.842$

$\Rightarrow \quad a - 131 = -45.468$

$\Rightarrow \quad a = -45.468 + 131 \quad \Rightarrow \quad a = 85.532 \approx 85.5$ g

$\therefore$ A small orange weighs at most 85.5 grams. **(Ans).**

Question 49

The residents of Mahjing were asked to classify their local bus service:

- 25% of residents classified their service as good.
- 60% of residents classified their service as satisfactory.
- 15% of residents classified their service as poor.

(a) A random sample of 110 residents of Mahjing is chosen.
Use a suitable approximation to find the probability that fewer than 22 residents classified their bus service as good. [5]

(b) For a random sample of 10 residents of Mahjing, find the probability that fewer than 8 classified their bus service as good or satisfactory. [3]

(c) Three residents of Mahjing are selected at random.
Find the probability that one resident classified the bus service as good, one as satisfactory and one as poor. [2]

[J24/P5/Q6]

Suggested Solution: **Learning CORNER**

(a) Total number of residents, $n = 110$

P(bus service is good), $p = 0.25$

P(bus service is not good), $q = 0.75$

$\Rightarrow$ $X \sim B(110,\ 0.25)$

Binomial distribution can be approximated as normal ditribution

Mean, $\mu = np = 110 \times 0.25 = 27.5$

Variance, $\sigma^2 = npq = 110 \times 0.25 \times 0.75 = 20.625$

Let X be a function of normal distribution, i.e. $X \sim N(27.5,\ 20.625)$

For $P(X < 22)$

Applying continuity correction,

$P(X < 21.5)$ $\qquad X \sim N(27.5,\ 20.625)$

$\Rightarrow P\left(\dfrac{X-\mu}{\sigma} < \dfrac{21.5 - 27.5}{\sqrt{20.625}}\right)$ $\qquad$ Standardizing X, using $\dfrac{X-\mu}{\sigma} = Z$

$\Rightarrow P(Z < -1.321)$ $\qquad \because\ Z \sim N(0,\ 1)$

$\Rightarrow \Phi(-1.321)$

$= 1 - \Phi(1.321)$

$= 1 - (0.9068) = 0.0932$ **(Ans).**

(b) Number of residents, $n = 10$

P(bus service is good or satisfactory) $= p = 0.25 + 0.60 = 0.85$

P(bus service is poor) $= q = 0.15$

Let X be a function of binomial distribution i.e. $X \sim B(10,\ 0.85)$

P(fewer than 8 classified their bus service good or satisfactory)

$= P(X < 8)$

$= P(X = 0,\ 1,\ 2,\ 3,\ .....\ 7)$

$= 1 - P(X = 8,\ 9,\ 10)$

$= 1 - \left[{}^{10}C_8 (0.85)^8 (0.15)^2 + {}^{10}C_9 (0.85)^9 (0.15)^1 + {}^{10}C_{10} (0.85)^{10} (0.15)^0\right]$

$= 1 - (0.275897 + 0.347425 + 0.196874)$

$= 1 - 0.820196$

$= 0.179804 \approx 0.180$ (3sf) **(Ans).**

(c) P(one classified as good, one satisfactory & one poor)

$= P$(GSP or GPS or PGS or PSG or SPG or SGP)

$= {}^3P_3\ P(\text{GSP})$

$= {}^3P_3[0.25 \times 0.60 \times 0.15]$

$= 0.135$ (3sf) **(Ans).**