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About A LEVEL

MATHEMATICS Paper 3

(TOPICAL)

period	2010 to 2025
contents	June & November, P3, Worked Solutions
form	Topic By Topic
compiled for	A Levels
special features	Learning Corner

About **Learning Corner**

This column will further enrich students with the extra information and knowledge provided. It improves the students' ability to solve problems of similar style. Other important guidances are also stated.

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New Syllabus

Revision

Learning CORNER

Topic 4
Logarithmic and Exponential Functions

Question 1

Solve the equation $\ln(2 + e^{-x}) = 2$, giving your answer correct to 2 decimal places. [4]

[J09/P3/Q1]

Suggested Solution:

$$\ln(2 + e^{-x}) = 2$$
$$2 + e^{-x} = e^2$$
$$e^{-x} = e^2 - 2$$
$$-x = \ln(e^2 - 2)$$
$$x = -\ln(e^2 - 2)$$
$$x = -\ln(5.389056)$$
$$x = -1.68 \quad \textbf{(Ans).}$$

Remember:

If $\ln x = u \Rightarrow x = e^u$

Conversely

If $x = e^u \Rightarrow u = \ln x$

Question 2

Solve the equation

$$\ln(5 - x) = \ln 5 - \ln x,$$

giving your answers correct to 3 significant figures. [4]

[N09/P3/Q1]

Suggested Solution:

$$\ln(5 - x) = \ln 5 - \ln x$$
$$\ln(5 - x) = \ln\left(\frac{5}{x}\right)$$
$$5 - x = \frac{5}{x}$$
$$5x - x^2 = 5$$
$$x^2 - 5x + 5 = 0$$

applying quadratic formula,

$$x = \frac{-(-5) \pm \sqrt{(-5)^2 - 4(1)(5)}}{2(1)} = \frac{5 \pm \sqrt{25 - 20}}{2} = \frac{5 \pm \sqrt{5}}{2}$$

$$\Rightarrow \quad x = \frac{5 + \sqrt{5}}{2} \quad \text{or} \quad x = \frac{5 - \sqrt{5}}{2}$$

$$\Rightarrow \quad x = 3.62 \text{ or } x = 1.38 \quad \textbf{(Ans).}$$

Note that:

- $\ln\left(\frac{a}{b}\right) = \ln a - \ln b$
- $\ln(a \times b) = \ln a + \ln b$
- $\ln(a^b) = b \ln a$

Question 3

Solve the equation $\dfrac{2^x + 1}{2^x - 1} = 5,$

giving your answer correct to 3 significant figures. [4]

[J10/P3/Q1]

Suggested Solution:

$$\frac{2^x+1}{2^x-1}=5$$

Let $2^x = y$

$$\Rightarrow \quad \frac{y+1}{y-1}=5$$

$$y+1=5(y-1)$$

$$4y=6$$

$$y=1.5$$

$$\Rightarrow \quad 2^x=1.5$$

$$\log(2^x)=\log(1.5)$$

$$x\log(2)=\log(1.5)$$

$$x=\frac{\log 1.5}{\log 2}=0.585 \quad \textbf{(Ans).}$$

Question 4

Solve the equation

$$\ln(1+x^2)=1+2\ln x,$$

giving your answer correct to 3 significant figures. [4]

[N10/P3/Q2]

Suggested Solution:

$$\ln(1+x^2)=1+2\ln x$$

$$\Rightarrow \quad \ln(1+x^2)-2\ln x=1 \quad \Rightarrow \quad \ln(1+x^2)-\ln x^2=1 \quad \Rightarrow \quad \ln\left(\frac{1+x^2}{x^2}\right)=1$$

$$\Rightarrow \quad \frac{1+x^2}{x^2}=e \quad \Rightarrow \quad 1+x^2=ex^2 \quad \Rightarrow \quad ex^2-x^2=1 \quad \Rightarrow \quad x^2(e-1)=1$$

$$\Rightarrow \quad x^2=\frac{1}{(e-1)} \quad \Rightarrow \quad x=\pm\frac{1}{\sqrt{e-1}}$$

$$\Rightarrow \quad x=0.763 \quad \text{or} \quad x=-0.763 \text{ (rejected)}$$

$$\therefore \quad x=0.763 \quad \textbf{(Ans).}$$

Recall:

- $\ln(\frac{a}{b})=\ln a-\ln b$
- $\ln(a\times b)=\ln a+\ln b$
- $\ln(a^b)=b\ln a$
- $\ln e=1$
- $\ln x=1 \quad \Rightarrow \quad x=e^1$
- Remember that log of a negative number is not defined.

Question 5

(i) Show that the equation

$$\log_2(x+5)=5-\log_2 x$$

can be written as a quadratic equation in x. [3]

(ii) Hence solve the equation

$$\log_2(x+5)=5-\log_2 x. \quad [2]$$

[J11/P3/Q2]

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Suggested Solution:

(i) $\log_2(x+5) = 5 - \log_2 x$

$\log_2(x+5) + \log_2 x = 5$

$\log_2 x(x+5) = 5$

$x(x+5) = 2^5$

$x^2 + 5x = 32$

$x^2 + 5x - 32 = 0$ **(Shown).**

(ii) $\log_2(x+5) = 5 - \log_2 x$

using the result of part (i), the equation can be written as,

$x^2 + 5x - 32 = 0$

applying quadratic formula

$$x = \frac{-5 \pm \sqrt{(5)^2 - 4(1)(-32)}}{2(1)} = \frac{-5 \pm \sqrt{25+128}}{2} = \frac{-5 \pm \sqrt{153}}{2}$$

$\Rightarrow \quad x = \dfrac{-5+\sqrt{153}}{2} \quad \text{or} \quad \dfrac{-5-\sqrt{153}}{2}$

$\Rightarrow \quad x = 3.68 \quad \text{or} \quad -8.68$

as log of negative number is not defined,

$\therefore \quad x = 3.68$ **(Ans).**

Question 6

Using the substitution $u = e^x$, or otherwise, solve the equation

$$e^x = 1 + 6e^{-x},$$

giving your answer correct to 3 significant figures. [4]

[N11/P3/Q1]

Suggested Solution:

$e^x = 1 + 6e^{-x} \quad \Rightarrow \quad e^x = 1 + \dfrac{6}{e^x}$

substitute $u = e^x$,

$\Rightarrow \quad u = 1 + \dfrac{6}{u} \quad \Rightarrow \quad u^2 = u + 6 \quad \Rightarrow \quad u^2 - u - 6 = 0$

$\Rightarrow \quad u^2 - 3u + 2u - 6 = 0 \quad \Rightarrow \quad u(u-3) + 2(u-3) = 0 \quad \Rightarrow \quad (u-3)(u+2) = 0$

$\Rightarrow \quad u = 3 \quad \text{or} \quad u = -2$

$\Rightarrow \quad e^x = 3 \quad \text{or} \quad e^x = -2$ (not possible)

taking $\ln$ on both sides

$\ln e^x = \ln 3 \quad \Rightarrow \quad x \ln e = \ln 3 \quad \Rightarrow \quad x = \ln 3 = 1.099 \approx 1.10$ **(Ans).**

Question 7

Solve the equation

$$\text{In}(3x+4) = 2\,\text{In}(x+1),$$

giving your answer correct to 3 significant figures. [4]

[J12/P3/Q1]

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Suggested Solution:

$$\ln(3x+4) = 2\ln(x+1)$$

$$\Rightarrow \quad \ln(3x+4) = \ln(x+1)^2$$

$$\Rightarrow \quad 3x+4 = (x+1)^2 \quad \Rightarrow \quad 3x+4 = x^2+2x+1 \quad \Rightarrow \quad x^2-x-3=0$$

using quadratic formula, we have,

$$x = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(-3)}}{2(1)}$$

$$= \frac{1 \pm \sqrt{13}}{2}$$

$$\Rightarrow \quad x = \frac{1+\sqrt{13}}{2} \quad \text{or} \quad x = \frac{1-\sqrt{13}}{2}$$

$$\Rightarrow \quad x = 2.30 \qquad x = -1.30 \text{ (ignored)}$$

$\therefore \quad x = 2.30$ **(Ans).**

Recall:

- $b \log_x a = \log_x(a^b)$
- If $\log_x a = \log_x b$
  $\Rightarrow \quad a = b$

Question 8

Solve the equation

$$5^{x-1} = 5^x - 5,$$

giving your answer correct to 3 significant figures. [4]

[N12/P3/Q2]

Suggested Solution:

$$5^{x-1} = 5^x - 5$$

$$\Rightarrow \quad 5^x \times 5^{-1} = 5^x - 5$$

let $5^x = y$

$$\Rightarrow \quad \frac{y}{5} = y-5 \quad \Rightarrow \quad y = 5y-25 \quad \Rightarrow \quad 4y = 25 \quad \Rightarrow \quad y = 6.25$$

i.e. $5^x = 6.25$

taking lg on both sides,

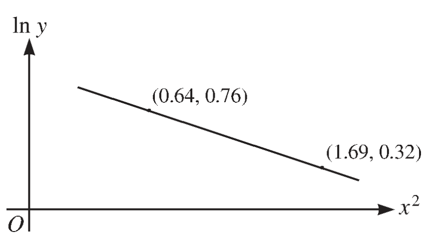
$$\lg 5^x = \lg 6.25$$

$$\Rightarrow \quad x\lg 5 = \lg 6.25 \quad \Rightarrow \quad x = \frac{\lg 6.25}{\lg 5} = 1.14$$ **(Ans).**

Note that:

- $\ln(a \times b) = \ln a + \ln b$
- $\ln(\frac{a}{b}) = \ln a - \ln b$
- $\ln(a^b) = b\ln a$
- $\ln e = 1$
- $\ln 1 = 0$

Question 9



The variables x and y satisfy the equation $y = Ae^{-kx^2}$, where A and k are constants. The graph of $\ln y$ against x^2 is a straight line passing through the points (0.64, 0.76) and (1.69, 0.32), as shown in the diagram. Find the values of A and k correct to 2 decimal places. [5]

[J13/P3/Q3]

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Suggested Solution:

$y = Ae^{-kx^2}$

Taking ln on both sides,

$\ln y = \ln(Ae^{-kx^2})$

$\Rightarrow \quad \ln y = \ln A + \ln e^{-kx^2} \quad \Rightarrow \quad \ln y = -kx^2 + \ln A$

$\therefore \quad$ gradient of the line $= -k$

from graph, using points (0.64, 0.76) and (1.69, 0.32),

$$\text{gradient} = \frac{0.76 - 0.32}{0.64 - 1.69} = -0.419$$

$\therefore \quad -k = -0.419 \quad \Rightarrow \quad k = 0.419 \approx 0.42$ **(Ans).**

for $\ln A$, substitute (0.64, 0.76) into the equation of line,

$0.76 = -0.419(0.64) + \ln A$

$0.76 = -0.2682 + \ln A$

$\ln A = 1.0282$

$A = 2.796 \approx 2.80$ **(Ans).**

Note that:

- $\ln(a \times b) = \ln a + \ln b$
- $\ln(a^b) = b \ln a$
- $\ln e = 1$

Question 10

Solve the equation $2|3^x - 1| = 3^x$, giving your answers correct to 3 significant figures. [4]

[N13/P3/Q2]

Suggested Solution:

$2|3^x - 1| = 3^x$

$\Rightarrow \quad 2(3^x - 1) = 3^x \quad$ or $\quad -2(3^x - 1) = 3^x$

$\Rightarrow \quad 2(3^x) - 2 = 3^x \quad$ or $\quad -2(3^x) + 2 = 3^x$

$\Rightarrow \quad 3^x = 2 \qquad 3^x = \frac{2}{3}$

taking lg on both sides,

$\lg 3^x = \lg 2 \quad$ or $\quad \lg 3^x = \lg \frac{2}{3}$

$\Rightarrow \quad x \lg 3 = \lg 2 \quad$ or $\quad x \lg 3 = \lg \frac{2}{3}$

$\Rightarrow \quad x = \frac{\lg 2}{\lg 3} \quad$ or $\quad x = \frac{\lg \frac{2}{3}}{\lg 3} \quad \Rightarrow \quad x = 0.631 \quad$ or $\quad x = -0.369$ **(Ans).**

Question 11

Solve the equation

$$2\ln(5 - e^{-2x}) = 1,$$

giving your answer correct to 3 significant figures. [4]

[J14/P3/Q2]

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Suggested Solution:

$2\ln(5-e^{-2x})=1$

$\Rightarrow \quad \ln(5-e^{-2x})=\dfrac{1}{2} \quad \Rightarrow \quad 5-e^{-2x}=e^{\frac{1}{2}} \quad \Rightarrow \quad e^{-2x}=5-e^{\frac{1}{2}}$

$\Rightarrow \quad \ln e^{-2x}=\ln(5-e^{\frac{1}{2}}) \quad \Rightarrow \quad -2x=1.2093 \quad \Rightarrow \quad x=-0.605$ **(Ans).**

Remember:

If $\ln x=u \quad \Rightarrow \quad x=e^u$

Conversely

If $x=e^u \quad \Rightarrow \quad u=\ln x$

Question 12

Use logarithms to solve the equation $e^x=3^{x-2}$, giving your answer correct to 3 decimal places. [3]

[N14/P3/Q1]

Suggested Solution:

$e^x=3^{x-2}$

taking ln on both sides

$\ln e^x=\ln 3^{x-2} \quad \Rightarrow \quad x\ln e=(x-2)\ln 3 \quad \Rightarrow \quad x=x\ln 3-2\ln 3$

$\Rightarrow \quad x\ln 3-x=2\ln 3 \quad \Rightarrow \quad x(\ln 3-1)=2\ln 3 \quad \Rightarrow \quad x=\dfrac{2\ln 3}{\ln 3-1}=22.281$ **(Ans).**

Note that:

- $\ln(a\times b)=\ln a+\ln b$
- $\ln(\frac{a}{b})=\ln a-\ln b$
- $\ln(a^b)=b\ln a$
- $\ln e=1$

Question 13

Using the substitution $u=4^x$, solve the equation $4^x+4^2=4^{x+2}$, giving your answer correct to 3 significant figures. [4]

[J15/P3/Q2]

Suggested Solution:

$4^x+4^2=4^{x+2}$

$\Rightarrow \quad 4^x+16=4^x\times 4^2 \quad \Rightarrow \quad 4^x+16=16(4^x)$

given substitution is, $\quad u=4^x$

$\Rightarrow \quad u+16=16u \quad \Rightarrow \quad 15u=16 \quad \Rightarrow \quad u=\dfrac{16}{15}$

$\therefore \quad 4^x=\dfrac{16}{15}$

taking lg on both sides

$\lg 4^x=\lg\dfrac{16}{15}$

$\Rightarrow \quad x\lg 4=\lg\dfrac{16}{15}$

$\Rightarrow \quad x=\lg\dfrac{16}{15}\div\lg 4 \;=\; 0.04655 \;\approx\; 0.0466$ (3sf) **(Ans).**

Question 14

Using the substitution $u = 3^x$, solve the equation $3^x + 3^{2x} = 3^{3x}$ giving your answer correct to 3 significant figures. [5]

[N15/P3/Q2]

Suggested Solution:

$3^x + 3^{2x} = 3^{3x} \quad \Rightarrow \quad 3^x + (3^x)^2 = (3^x)^3$

substitute $u = 3^x$,

$\Rightarrow \quad u + u^2 = u^3 \quad \Rightarrow \quad u^3 - u^2 - u = 0 \quad \Rightarrow \quad u(u^2 - u - 1) = 0$

$\Rightarrow \quad u = 0$ or $u^2 - u - 1 = 0$

$\Rightarrow \quad 3^x = 0$ (no solution) using quadratic formula

$$u = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(-1)}}{2(1)}$$

$$\Rightarrow \quad u = \frac{1 \pm \sqrt{5}}{2}$$

$$\Rightarrow \quad u = \frac{1+\sqrt{5}}{2} \quad \text{or} \quad u = \frac{1-\sqrt{5}}{2}$$

$$= 1.618 \quad \text{or} \quad = -0.618$$

$\Rightarrow \quad 3^x = 1.618$ or $3^x = -0.618$ (no solution)

taking lg on both sides

$\lg 3^x = \lg(1.618)$

$x \lg 3 = \lg(1.618)$

$x = \dfrac{\lg(1.618)}{\lg 3} = 0.438$ (3sf) **(Ans).**

Question 15

Use logarithms to solve the equation $4^{3x-1} = 3(5^x)$, giving your answer correct to 3 decimal places. [4]

[J16/P3/Q1]

Suggested Solution:

$4^{3x-1} = 3(5)^x$

$\Rightarrow \quad 4^{3x} \times 4^{-1} = 3 \times (5)^x \quad \Rightarrow \quad (4^3)^x \times \dfrac{1}{4} = 3 \times (5)^x \quad \Rightarrow \quad \left(\dfrac{64}{5}\right)^x = 12 \quad \Rightarrow \quad (12.8)^x = 12$

taking lg on both sides,

$\lg(12.8)^x = \lg 12 \quad \Rightarrow \quad x \lg(12.8) = \lg 12 \quad \Rightarrow \quad x = \dfrac{\lg 12}{\lg 12.8} = 0.975$ **(Ans).**

Question 16

Solve the equation $\dfrac{3^x + 2}{3^x - 2} = 8$, giving your answer correct to 3 decimal places. [3]

[N16/P3/Q1]

Suggested Solution:

$\dfrac{3^x + 2}{3^x - 2} = 8$

$3^x + 2 = 8(3^x - 2)$

$3^x + 2 = 8(3^x) - 16$

$3^x - 8(3^x) = -18$

$3^x = \dfrac{18}{7}$

$\lg 3^x = \lg(\dfrac{18}{7}) \quad \Rightarrow \quad x = \dfrac{\lg(\frac{18}{7})}{\lg 3} = 0.85968 \approx 0.860$ **(Ans).**

Learning CORNER

Question 17

Solve the equation $\ln(x^2+1) = 1 + 2\ln x$, giving your answer correct to 3 significant figures. [3]

[J17/P3/Q1]

Suggested Solution:

Note that:

- $\ln(a \times b) = \ln a + \ln b$
- $\ln(a^b) = b\ln a$
- $\ln e = 1$

$\ln(x^2+1) = 1 + 2\ln x$

$\Rightarrow \ln(x^2+1) = \ln e + \ln x^2$

$\Rightarrow \ln(x^2+1) = \ln(ex^2)$

$\Rightarrow x^2 + 1 = ex^2$

$\Rightarrow x^2(e-1) = 1 \Rightarrow x^2 = \dfrac{1}{e-1} \Rightarrow x^2 = 0.581977 \Rightarrow x = \pm 0.763$

$\Rightarrow x = 0.763$ or $x = -0.763$ (invalid as ln of -ve number is not defined)

$\therefore x = 0.763$ **(Ans).**

Question 18

Showing all necessary working, solve the equation $2\log_2 x = 3 + \log_2(x+1)$, giving your answer correct to 3 significant figures. [5]

[N17/P3/Q2]

Suggested Solution:

Recall:

- $b \log_x a = \log_x(a^b)$
- If $\log_x a = \log_x b$ $\Rightarrow a = b$

$2\log_2 x = 3 + \log_2(x+1)$

$\log_2 x^2 = 3\log_2 2 + \log_2(x+1)$

$\log_2 x^2 = \log_2 2^3 + \log_2(x+1)$

$\log_2 x^2 = \log_2\left(2^3(x+1)\right)$

$x^2 = 8x + 8 \Rightarrow x^2 - 8x - 8 = 0$

using quadratic formula,

$$x = \frac{-(-8) \pm \sqrt{(-8)^2 - 4(1)(-8)}}{2(1)} = \frac{8 \pm \sqrt{96}}{2}$$

$\Rightarrow x = \dfrac{8+\sqrt{96}}{2}$ or $x = \dfrac{8-\sqrt{96}}{2}$

$\Rightarrow x = 8.899$ $x = -0.899$ (ignored)

$\therefore x = 8.90$ **(Ans).**

Question 19

Showing all necessary working, solve the equation $3\left|2^x - 1\right| = 2^x$, giving your answers correct to 3 significant figures. [4]

[J18/P3/Q1]

Suggested Solution:

$3\left|2^x-1\right|=2^x$

Let $2^x=y \Rightarrow 3|y-1|=y$

squaring both sides,

$\Rightarrow (3|y-1|)^2=y^2$

$\Rightarrow 9(y^2-2y+1)=y^2$

$\Rightarrow 8y^2-18y+9=0 \quad \Rightarrow \quad (2y-3)(4y-3)=0$

$\therefore \quad y=1.5 \quad$ or $\quad y=0.75$

$\Rightarrow 2^x=1.5 \quad$ or $\quad 2^x=0.75$

taking lg on both sides,

$\lg 2^x=\lg 1.5 \quad$ or $\quad \lg 2^x=\lg 0.75$

$x\lg 2=\lg 1.5 \quad$ or $\quad x\lg 2=\lg 0.75$

$x=\dfrac{\lg 1.5}{\lg 2} \quad$ or $\quad x=\dfrac{\lg 0.75}{\lg 2}$

$x=0.585 \quad$ or $\quad x=-0.415$

$\therefore \quad x=0.585,\ -0.415$ **(Ans).**

Question 20

Showing all necessary working, solve the equation

$$\frac{e^x+e^{-x}}{e^x+1}=4,$$

giving your answer correct to 3 decimal places. [5]

[N18/P3/Q4]

Suggested Solution:

$\dfrac{e^x+e^{-x}}{e^x+1}=4$

$\Rightarrow e^x+e^{-x}=4e^x+4 \quad \Rightarrow \quad e^x+\dfrac{1}{e^x}=4e^x+4$

Let $e^x=y$

$\Rightarrow y+\dfrac{1}{y}=4y+4 \quad \Rightarrow \quad y^2+1=4y^2+4y \quad \Rightarrow \quad 3y^2+4y-1=0$

Using quadratic formula

$y=\dfrac{-4\pm\sqrt{(4)^2-4(3)(-1)}}{2(3)}$

$\Rightarrow y=\dfrac{-4\pm\sqrt{28}}{6}$

$\Rightarrow y=\dfrac{-4+\sqrt{28}}{6} \quad$ or $\quad y=\dfrac{-4-\sqrt{28}}{6}$

$\therefore \quad y=0.2153 \quad$ or $\quad y=-1.549$

$\Rightarrow e^x=0.2153 \qquad e^x=-1.549$ (ignored)

$\Rightarrow x=\ln(0.2153)$

$=-1.536$ **(Ans).**

Learning CORNER

> ***Question 29***
>
> Solve the equation $\ln(2x^2-3) = 2\ln x - \ln 2$, giving your answer in an exact form. [3]
>
> *[J23/P3/Q2]*

Suggested Solution:

$\ln(2x^2-3) = 2\ln x - \ln 2$

$\Rightarrow \ln(2x^2-3) = \ln x^2 - \ln 2$

$\Rightarrow \ln(2x^2-3) = \ln\left(\dfrac{x^2}{2}\right)$

$\Rightarrow 2x^2 - 3 = \dfrac{x^2}{2}$

$\Rightarrow 4x^2 - 6 = x^2$

$\Rightarrow 3x^2 = 6 \Rightarrow x^2 = 2 \Rightarrow x = \pm\sqrt{2}$

$x = -\sqrt{2}$ is rejected as $\ln$ of negative number is not defined

$\therefore \quad x = \sqrt{2}$ **(Ans).**

> ***Question 30***
>
> The variables x and y satisfy the equation $a^{2y-1} = b^{x-y}$, where a and b are constants.
>
> **(a)** Show that the graph of y against x is a straight line. [3]
>
> **(b)** Given that $a = b^3$, state the equation of the straight line in the form $y = px + q$, where p and q are rational numbers in their simplest form. [2]
>
> *[J24/P3/Q3]*

Suggested Solution:

(a) $a^{2y-1} = b^{x-y}$,

Taking $\ln$ on both sides

$\ln a^{2y-1} = \ln b^{x-y}$

$\Rightarrow (2y-1)\ln a = (x-y)\ln b$

$\Rightarrow 2y\ln a - \ln a = x\ln b - y\ln b$

$\Rightarrow 2y\ln a + y\ln b = x\ln b + \ln a$

$y(2\ln a + \ln b) = (\ln b)x + \ln a$

$\Rightarrow y = \dfrac{\ln b}{2\ln a + \ln b}x + \dfrac{\ln a}{2\ln a + \ln b}$

Which is of the form $y = mx + c$.

Thus, the graph of y against x is a straight-line. **(Shown).**

Remember:

- $\ln(a \times b) = \ln a + \ln b$
- $\ln(\frac{a}{b}) = \ln a - \ln b$
- $\ln(a^b) = b\ln a$
- $\ln e = 1$
- $\ln 1 = 0$

(b) $a^{2y-1} = b^{x-y}$

Given that, $a = b^3$

$\Rightarrow \ (b^3)^{2y-1} = b^{x-y}$

$\Rightarrow \ b^{6y-3} = b^{x-y}$

$\Rightarrow \ 6y-3 = x-y$

$\Rightarrow \ 7y = x+3 \ \Rightarrow \ y = \frac{1}{7}x + \frac{3}{7}$ **(Ans).**

Question 31

Solve the equation $5^x = 5^{x+2} - 10$. Give your answer correct to 3 decimal places. [3]

[N24/P3/Q4]

Suggested Solution:

$5^x = 5^{x+2} - 10$

$\Rightarrow \ 5^x = (5^x \times 5^2) - 10$

Let $y = 5^x$

$\Rightarrow \ y = (y \times 5^2) - 10$

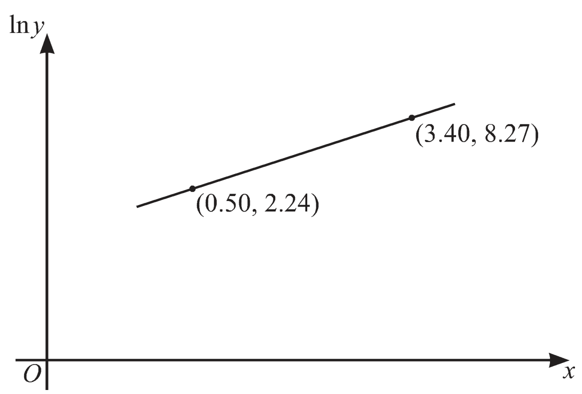
$\Rightarrow \ y = 25y - 10 \ \Rightarrow \ 24y = 10 \ \Rightarrow \ y = \frac{10}{24} \ \Rightarrow \ y = \frac{5}{12}$

$\therefore \ 5^x = \frac{5}{12}$

$\Rightarrow \ \ln 5^x = \ln\left(\frac{5}{12}\right)$

$\Rightarrow \ x \ln 5 = \ln\left(\frac{5}{12}\right) \ \Rightarrow \ x = \frac{\ln \frac{5}{12}}{\ln 5} \ \Rightarrow \ x = -0.544$ **(Ans).**

Question 32



The variables x and y satisfy the equation $ay = b^x$, where a and b are constants. The graph of $\ln y$ against x is a straight line passing through the points (0.50, 2.24) and (3.40, 8.27), as shown in the diagram.

Find the values of a and b. Give each value correct to 1 significant figure. [4]

[N24/P3/Q6]

Suggested Solution:

$ay = b^x$

Taking ln on both sides,

$\ln ay = \ln b^x$

$\Rightarrow \ln a + \ln y = x \ln b \quad \Rightarrow \quad \ln y = (\ln b)x - \ln a$ .... (1)

Using $(0.50, 2.24)$ and $(3.40, 8.27)$,

Gradient $= \dfrac{8.27 - 2.24}{3.40 - 0.50} = \dfrac{6.03}{2.9} = 2.0793$

$\Rightarrow \ln b = 2.0793 \quad \Rightarrow \quad b = e^{2.0793} \quad \Rightarrow \quad b = 7.99 = 8$

Substitute $(0.50, 2.24)$ into (1),

$2.24 = (2.0793)(0.50) - \ln a$

$\Rightarrow 2.24 = 1.03965 - \ln a$

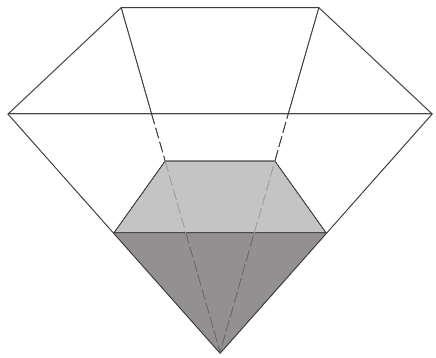
$\Rightarrow \ln a = 1.03965 - 2.24 \quad \Rightarrow \quad \ln a = -1.20 \quad \Rightarrow \quad a = 0.30$

$\therefore \quad a = 0.3, \quad b = 8$ **(Ans).**

Learning CORNER

Topic 10
Differential Equations

Question 1



An underground storage tank is being filled with liquid as shown in the diagram. Initially the tank is empty. At time t hours after filling begins, the volume of liquid is $V\,\mathrm{m}^3$ and the depth of liquid is h m. It is given that $V = \frac{4}{3}h^3$.

The liquid is poured in at a rate of $20\,\mathrm{m}^3$ per hour, but owing to leakage, liquid is lost at a rate proportional to h^2. when $h = 1$, $\frac{\mathrm{d}h}{\mathrm{d}t} = 4.95$.

(i) Show that h satisfies the differential equation

$$\frac{\mathrm{d}h}{\mathrm{d}t} = \frac{5}{h^2} - \frac{1}{20}.$$ [4]

(ii) Verify that $\frac{20h^2}{100-h^2} \equiv -20 + \frac{2000}{(10-h)(10+h)}$. [1]

(iii) Hence solve the differential equation in part (**i**), obtaining an expression for t in terms of h. [5]

[N08/P3/Q8]

Suggested Solution:

(i) Given that, $\frac{dV}{dt} = 20$ ………(i) and $\frac{dV}{dt} \propto -h^2 \;\Rightarrow\; \frac{dV}{dt} = -kh^2$ ………(ii)

combining equations (i) and (ii),

$$\frac{dV}{dt} = 20 - kh^2$$

also given, $V = \frac{4}{3}h^3$

differentiating w.r.t h, $\frac{dV}{dh} = \frac{4}{3}(3h^2) = 4h^2$

now, $$\frac{dh}{dt} = \frac{dh}{dV} \times \frac{dV}{dt}$$
$$= \left(\frac{1}{4h^2}\right) \times (20 - kh^2)$$
$$= \frac{5}{h^2} - \frac{k}{4} \quad \cdots\cdots\cdots\text{(iii)}$$

given that, when $h=1$, $\dfrac{dh}{dt}=4.95$

$\Rightarrow \quad 4.95=\dfrac{5}{1^2}-\dfrac{k}{4} \quad \Rightarrow \quad 4.95=5-\dfrac{k}{4} \quad \Rightarrow \quad \dfrac{k}{4}=0.05 \quad \Rightarrow \quad k=0.2$

$\therefore$ equation (iii) becomes,

$$\frac{dh}{dt}=\frac{5}{h^2}-\frac{0.2}{4} \quad \Rightarrow \quad \frac{dh}{dt}=\frac{5}{h^2}-\frac{1}{20} \quad \textbf{(Shown).}$$

(ii) R.H.S. $=-20+\dfrac{2000}{(10+h)(10-h)}$

$$=-20+\frac{2000}{100-h^2}$$

$$=\frac{-20(100-h^2)+2000}{100-h^2}$$

$$=\frac{-2000+20h^2+2000}{(100-h^2)}=\frac{20h^2}{100-h^2}=\text{L.H.S.} \quad \textbf{(Verified).}$$

(iii) From part (i),

$$\frac{dh}{dt}=\frac{5}{h^2}-\frac{1}{20}$$

$$\Rightarrow \quad \frac{dh}{dt}=\frac{100-h^2}{20h^2} \quad \Rightarrow \quad \frac{20h^2}{100-h^2}\,dh=dt$$

using the result of part (ii),

$$\left(-20+\frac{2000}{(10-h)(10+h)}\right)dh=dt$$

integrating both sides

$$\int\left(-20+\frac{2000}{(10-h)(10+h)}\right)dh=\int dt$$

$$\int\left(-20+2000\left(\frac{1}{(10^2-h^2)}\right)\right)dh=\int dt$$

$$-20h+2000\left(\frac{1}{2(10)}\ln(\frac{10+h}{10-h})\right)=t+C$$

$$-20h+100\ln(\frac{10+h}{10-h})=t+C$$

when $t=0$, $h=0$

$$\Rightarrow \quad -20(0)+100\ln(\frac{10+0}{10-0})=0+C \quad \Rightarrow \quad 100\ln(1)=C \quad \Rightarrow \quad C=0$$

$$\therefore \quad -20h+100\ln(\frac{10+h}{10-h})=t \quad \textbf{(Ans).}$$

Note that:

$$\int\frac{1}{a^2-x^2}\,dx=\frac{1}{2a}\ln\left(\frac{a+x}{a-x}\right)+K$$

Question 2

The variables x and t are related by the differential equation

$$e^{2t}\frac{\mathrm{d}x}{\mathrm{d}t}=\cos^2 x,$$

where $t \geq 0$. When $t=0$, $x=0$.

(i) Solve the differential equation, obtaining an expression for x in terms of t. [6]

(ii) State what happens to the value of x when t becomes very large. [1]

(iii) Explain why x increases as t increases. [1]

[J10/P3/Q7]

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Suggested Solution:

(i) $e^{2t}\dfrac{dx}{dt}=\cos^2 x$

Separating the variables, we have,

$$\frac{1}{\cos^2 x}\,dx=\frac{1}{e^{2t}}\,dt \quad\Rightarrow\quad \sec^2 x\,dx=e^{-2t}\,dt$$

integrating both sides

$$\int \sec^2 x\,dx=\int e^{-2t}\,dt$$

$$\Rightarrow \qquad \tan x=\frac{e^{-2t}}{-2}+K$$

when $t=0,\ x=0$

$$\Rightarrow \quad \tan(0)=-\frac{1}{2}e^{-2(0)}+K \quad\Rightarrow\quad 0=-\frac{1}{2}+K \quad\Rightarrow\quad K=\frac{1}{2}$$

$$\therefore \quad \tan x=-\frac{1}{2}e^{-2t}+\frac{1}{2}$$

$$\Rightarrow \quad \tan x=\frac{1}{2}(1-e^{-2t})$$

$$\Rightarrow \qquad x=\tan^{-1}\left(\frac{1}{2}(1-e^{-2t})\right) \quad \textbf{(Ans).}$$

(ii) As $t\to\infty$

$$e^{-2t}=\frac{1}{e^{2t}}\to 0$$

$$\Rightarrow \quad x\to\tan^{-1}(\frac{1}{2})$$

$\therefore$ when t becomes very large, x approaches to $\tan^{-1}(\frac{1}{2})$ **(Ans).**

(iii) $e^{2t}\dfrac{dx}{dt}=\cos^2 x$

$$\Rightarrow \quad \frac{dx}{dt}=e^{-2t}\cos^2 x\ >0, \quad \text{for all values of } t$$

$\Rightarrow$ x is an increasing function.

$\therefore$ x increases as t increases.

Question 3

A certain substance is formed in a chemical reaction. The mass of substance formed t seconds after the start of the reaction is x grams. At any time the rate of formation of the substance is proportional to $(20-x)$. When $t=0$, $x=0$ and $\dfrac{dx}{dt}=1$.

(i) Show that x and t satisfy the differential equation

$$\frac{dx}{dt}=0.05(20-x). \qquad [2]$$

(ii) Find, in any form, the solution of this differential equation. [5]

(iii) Find x when $t=10$, giving your answer correct to 1 decimal place. [2]

(iv) State what happens to the value of x as t becomes very large. [1]

[N10/P3/Q10]

Suggested Solution:

(i) According to given information,

$$\frac{dx}{dt} \propto (20-x) \quad \Rightarrow \quad \frac{dx}{dt} = k(20-x) \text{ ....(1)}$$

given that, $x=0$, $\frac{dx}{dt}=1$, we have,

$$1 = k(20-0) \quad \Rightarrow \quad 20k = 1 \quad \Rightarrow \quad k = \frac{1}{20}$$

$\therefore$ equation (1) becomes,

$$\frac{dx}{dt} = \frac{1}{20}(20-x) \quad \Rightarrow \quad \frac{dx}{dt} = 0.05(20-x) \quad \textbf{(Shown).}$$

(ii) $\frac{dx}{dt} = \frac{1}{20}(20-x) \quad \Rightarrow \quad \frac{1}{20-x}\,dx = \frac{1}{20}\,dt$

integrating both sides

$$\int \frac{1}{20-x}\,dx = \frac{1}{20}\int dt \quad \Rightarrow \quad -\int \frac{-1}{20-x}\,dx = \frac{1}{20}\int dt \quad \Rightarrow \quad -\ln|20-x| = \frac{1}{20}t + C$$

when $t=0$, $x=0$

$$\Rightarrow \quad -\ln|20-0| = \frac{1}{20}(0)t + C \quad \Rightarrow \quad C = -\ln 20$$

$$\therefore \quad -\ln|20-x| = \frac{1}{20}t - \ln 20 \quad \Rightarrow \quad \ln 20 - \ln|20-x| = \frac{1}{20}t \quad \Rightarrow \quad \ln\left|\frac{20}{20-x}\right| = \frac{1}{20}t$$

$$\Rightarrow \quad \frac{20}{20-x} = e^{\frac{1}{20}t} \quad \Rightarrow \quad 20e^{-\frac{1}{20}t} = 20-x \quad \Rightarrow \quad x = 20-20e^{-\frac{1}{20}t} \quad \textbf{(Ans).}$$

(iii) When $t=10$

$$x = 20-20e^{-\frac{1}{20}(10)} = 20-20e^{-\frac{1}{2}} = 20-12.131 = 7.869 \approx 7.9 \text{ (1 dp)} \quad \textbf{(Ans).}$$

(iv) $x = 20-20e^{-\frac{1}{20}t}$

As $t \to \infty$, $e^{-\frac{1}{20}t} \to 0$ and $x \to 20$

$\therefore$ x approaches to 20 as t becomes very large. **(Ans).**

Question 4

A certain curve is such that its gradient at a point (x, y) is proportional to xy. At the point (1, 2) the gradient is 4.

(i) By setting up and solving a differential equation, show that the equation of the curve is $y = 2e^{x^2-1}$. [7]

(ii) State the gradient of the curve at the point (–1, 2) and sketch the curve. [2]

[J11/P3/Q6]

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Suggested Solution:

(i) $\dfrac{dy}{dx} \propto xy \quad \Rightarrow \quad \dfrac{dy}{dx} = kxy$

Given that, $\dfrac{dy}{dx} = 4$ at point (1, 2),

$\Rightarrow \quad 4 = k(1)(2) \quad \Rightarrow \quad 2k = 4 \quad \Rightarrow \quad k = 2$

$\therefore$ differential equation becomes

$$\frac{dy}{dx} = 2xy \quad \Rightarrow \quad \frac{1}{y}\,dy = 2x\,dx$$

integrating both sides

$$\int \frac{1}{y}\,dy = 2\int x\,dx$$

$\Rightarrow \quad \ln y = 2(\frac{x^2}{2}) + C \quad \Rightarrow \quad \ln y = x^2 + C$

the curve passes through (1, 2),

$\Rightarrow \quad \ln 2 = (1)^2 + C \quad \Rightarrow \quad C = \ln 2 - 1$

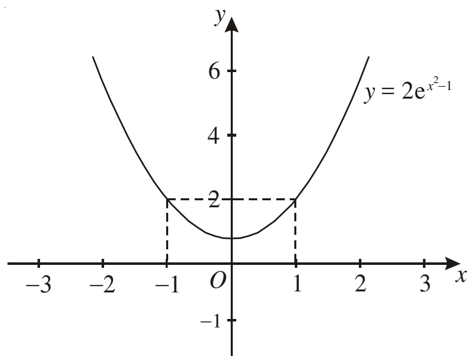
$\therefore \quad \ln y = x^2 + \ln 2 - 1$

$\Rightarrow \quad \ln y - \ln 2 = x^2 - 1$

$\Rightarrow \quad \ln\left(\dfrac{y}{2}\right) = x^2 - 1 \quad \Rightarrow \quad \dfrac{y}{2} = e^{x^2-1} \quad \Rightarrow \quad y = 2e^{x^2-1}$ **(Shown).**

(ii) From part **(i)**, $\dfrac{dy}{dx} = 2xy$

at $(-1, 2)$, gradient, $\dfrac{dy}{dx} = 2(-1)(2) = -4$ **(Ans).**



Remember:

If putting $x = -x$ in the equation does not change the equation, then curve is symmetric about y-axis and vice versa.

Question 5

The variables x and θ are related by the differential equation

$$\sin 2\theta \frac{dx}{d\theta} = (x+1)\cos 2\theta,$$

where $0 < \theta < \frac{1}{2}\pi$. When $\theta = \frac{1}{12}\pi$, $x = 0$. Solve the differential equation, obtaining an expression for x in terms of θ, and simplifying your answer as far as possible. [7]

[N11/P3/Q4]

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Suggested Solution:

$$\sin 2\theta \frac{dx}{d\theta} = (x+1)\cos 2\theta \quad \Rightarrow \quad \frac{1}{x+1}\,dx = \frac{\cos 2\theta}{\sin 2\theta}\,d\theta$$

Integrating both sides,

$$\int \frac{1}{x+1}\,dx = \int \frac{\cos 2\theta}{\sin 2\theta}\,d\theta$$

$$\Rightarrow \quad \int \frac{1}{x+1}\,dx = \frac{1}{2}\int \frac{2\cos 2\theta}{\sin 2\theta}\,d\theta \quad \Rightarrow \quad \ln|x+1| = \frac{1}{2}\ln|\sin 2\theta| + K \cdots\cdots\cdots(1)$$

when $\theta = \frac{\pi}{12}$, $x = 0$

$$\Rightarrow \quad \ln|0+1| = \frac{1}{2}\ln\left|\sin 2(\frac{\pi}{12})\right| + K \quad \Rightarrow \quad \ln 1 = \frac{1}{2}\ln\left|\sin\frac{\pi}{6}\right| + K \quad \Rightarrow \quad 0 = \frac{1}{2}\ln\left|\frac{1}{2}\right| + K$$

$$\Rightarrow \quad 0 = \frac{1}{2}(\ln 1 - \ln 2) + K \quad \Rightarrow \quad 0 = \frac{1}{2}(-\ln 2) + K \quad \Rightarrow \quad K = \frac{1}{2}\ln 2$$

$\therefore$ equation (1) becomes,

$$\ln|x+1| = \frac{1}{2}\ln|\sin 2\theta| + \frac{1}{2}\ln 2$$

$$\Rightarrow \quad \ln|x+1| = \frac{1}{2}\ln|2\sin 2\theta|$$

$$\Rightarrow \quad \ln|x+1| = \ln(2\sin 2\theta)^{\frac{1}{2}} \quad \Rightarrow \quad x+1 = \sqrt{2\sin 2\theta} \quad \Rightarrow \quad x = \sqrt{2\sin 2\theta} - 1 \quad \textbf{(Ans).}$$

Note that:

- $\ln(a \times b) = \ln a + \ln b$
- $\ln(\frac{a}{b}) = \ln a - \ln b$
- $\ln(a^b) = b\ln a$
- $\ln e = 1$
- $\ln 1 = 0$

Question 6

The variables x and y satisfy the differential equation

$$\frac{dy}{dx} = e^{2x+y},$$

and $y = 0$ when $x = 0$. Solve the differential equation, obtaining an expression for y in terms of x. [6]

[J12/P3/Q5]

Suggested Solution:

$$\frac{dy}{dx} = e^{2x+y}$$

$$\Rightarrow \quad \frac{dy}{dx} = e^{2x} \times e^{y} \quad \Rightarrow \quad \frac{1}{e^{y}}\,dy = e^{2x}\,dx \quad \Rightarrow \quad e^{-y}\,dy = e^{2x}\,dx$$

integrating both sides,

$$\int e^{-y}\,dy = \int e^{2x}\,dx$$

$$\Rightarrow \quad -e^{-y} = \frac{1}{2}e^{2x} + K \cdots\cdots\cdots(1)$$

when $x = 0$, $y = 0$

$$\Rightarrow \quad -e^{0} = \frac{1}{2}e^{0} + K \quad \Rightarrow \quad -1 = \frac{1}{2} + K \quad \Rightarrow \quad K = -\frac{3}{2}$$

$\therefore$ equation (1) becomes

$$-e^{-y} = \frac{1}{2}e^{2x} - \frac{3}{2}$$

$$\Rightarrow \quad -e^{-y} = \frac{e^{2x} - 3}{2} \quad \Rightarrow \quad -2e^{-y} = e^{2x} - 3 \quad \Rightarrow \quad 2e^{-y} = 3 - e^{2x}$$

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taking ln on both sides,

$$\ln(2e^{-y}) = \ln(3 - e^{2x})$$

$$\Rightarrow \quad \ln 2 + \ln(e^{-y}) = \ln(3 - e^{2x})$$

$$\Rightarrow \quad \ln 2 - y \ln e = \ln(3 - e^{2x})$$

$$\Rightarrow \quad -y = \ln(3 - e^{2x}) - \ln 2$$

$$\Rightarrow \quad y = \ln 2 - \ln(3 - e^{2x}) \quad \Rightarrow \quad y = \ln\left(\frac{2}{3 - e^{2x}}\right) \quad \textbf{(Ans).}$$

Recall:

- $\ln(\frac{a}{b}) = \ln a - \ln b$
- $\ln(a \times b) = \ln a + \ln b$
- $\ln(a^b) = b \ln a$
- $\ln e = 1$

Question 7

The variables x and y are related by the differential equation

$$x\frac{dy}{dx} = 1 - y^2.$$

When $x = 2$, $y = 0$. Solve the differential equation, obtaining an expression for y in terms of x. [8]

[N12/P3/Q6]

Suggested Solution:

$$x\frac{dy}{dx} = 1 - y^2 \quad \Rightarrow \quad \frac{1}{1 - y^2}\,dy = \frac{1}{x}\,dx \quad \Rightarrow \quad \frac{1}{(1+y)(1-y)}\,dy = \frac{1}{x}\,dx \quad \cdots\cdots\cdots(1)$$

expressing $\frac{1}{(1+y)(1-y)}$ in partial fractions, we have,

$$\frac{1}{(1+y)(1-y)} \equiv \frac{A}{1+y} + \frac{B}{1-y}$$

$$\Rightarrow \quad 1 \equiv A(1-y) + B(1+y)$$

for $y = 1$, $\quad 1 = A(1-1) + B(1+1) \quad \Rightarrow \quad 1 = 2B \quad \Rightarrow \quad B = \frac{1}{2}$

for $y = -1$, $\quad 1 = A\big(1-(-1)\big) + B(1-1) \quad \Rightarrow \quad 1 = 2A \quad \Rightarrow \quad A = \frac{1}{2}$

$$\therefore \quad \frac{1}{(1+y)(1-y)} = \frac{1}{2(1+y)} + \frac{1}{2(1-y)}$$

equation (1) can now be written as,

$$\left(\frac{1}{2(1+y)} + \frac{1}{2(1-y)}\right)dy = \frac{1}{x}\,dx$$

integration both sides,

$$\frac{1}{2}\int\left(\frac{1}{1+y} + \frac{1}{1-y}\right)dy = \int \frac{1}{x}\,dx$$

$$\Rightarrow \quad \frac{1}{2}\Big(\ln(1+y) - \ln(1-y)\Big) = \ln x + K \quad \Rightarrow \quad \frac{1}{2}\ln\left(\frac{1+y}{1-y}\right) = \ln x + K$$

given that, when $x = 2$, $y = 0$

$$\Rightarrow \quad \frac{1}{2}\ln\left(\frac{1+0}{1-0}\right) = \ln 2 + K$$

$$\Rightarrow \quad \frac{1}{2}\ln(1) = \ln 2 + K \quad \Rightarrow \quad 0 = \ln 2 + K \quad \Rightarrow \quad K = -\ln 2$$

Note that:

- $\ln(a \times b) = \ln a + \ln b$
- $\ln(\frac{a}{b}) = \ln a - \ln b$
- $\ln(a^b) = b \ln a$
- $\ln e = 1$
- $\ln 1 = 0$

$\therefore \quad \frac{1}{2}\ln\left(\frac{1+y}{1-y}\right) = \ln x - \ln 2$

$\Rightarrow \quad \ln\left(\frac{1+y}{1-y}\right) = 2\ln\left(\frac{x}{2}\right) \quad \Rightarrow \quad \ln\left(\frac{1+y}{1-y}\right) = \ln\left(\frac{x}{2}\right)^2$

$\Rightarrow \quad \frac{1+y}{1-y} = \frac{x^2}{4} \quad \Rightarrow \quad 4+4y = x^2 - x^2y \quad \Rightarrow \quad x^2y + 4y = x^2 - 4$

$\Rightarrow \quad y(x^2+4) = x^2 - 4 \quad \Rightarrow \quad y = \frac{x^2-4}{x^2+4}$ **(Ans).**

Question 8

(i) Express $\frac{1}{x^2(2x+1)}$ in the form $\frac{A}{x^2} + \frac{B}{x} + \frac{C}{2x+1}$. [4]

(ii) The variables x and y satisfy the differential equation

$$y = x^2(2x+1)\frac{\mathrm{d}y}{\mathrm{d}x},$$

and $y = 1$ when $x = 1$. Solve the differential equation and find the exact value of y when $x = 2$. Give your value of y in a form not involving logarithms. [7]

[J13/P3/Q8]

Suggested Solution:

(i) $\frac{1}{x^2(2x+1)}$

resolving into partial fraction

$$\frac{1}{x^2(2x+1)} \equiv \frac{A}{x^2} + \frac{B}{x} + \frac{C}{2x+1}$$

$\Rightarrow \quad 1 \equiv A(2x+1) + Bx(2x+1) + Cx^2$

for $x = 0$

$1 = A\big(2(0)+1\big) + B(0)\big(2(0)+1\big) + C(0)^2 \quad \Rightarrow \quad A = 1$

for $x = -\frac{1}{2}$

$1 = A\left(2(-\frac{1}{2})+1\right) + B(-\frac{1}{2})\left(2(-\frac{1}{2})+1\right) + C(-\frac{1}{2})^2$

$\Rightarrow \quad 1 = A(0) + B(0) + C(\frac{1}{4}) \quad \Rightarrow \quad 1 = \frac{1}{4}C \quad \Rightarrow \quad C = 4$

for $x = 1$

$1 = A\big(2(1)+1\big) + B(1)\big(2(1)+1\big) + C(1)^2$

$\Rightarrow \quad 1 = 3A + 3B + C \quad \Rightarrow \quad 1 = 3(1) + 3B + 4 \quad \Rightarrow \quad 3B = -6 \quad \Rightarrow \quad B = -2$

$\therefore \quad \frac{1}{x^2(2x+1)} \equiv \frac{1}{x^2} - \frac{2}{x} + \frac{4}{2x+1}$ **(Ans).**

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(ii) $y = x^2(2x+1)\dfrac{dy}{dx} \Rightarrow \dfrac{dy}{dx} = \dfrac{y}{x^2(2x+1)} \Rightarrow \dfrac{1}{y}\,dy = \dfrac{1}{x^2(2x+1)}\,dx$

integrating both sides,

$$\int \frac{1}{y}\,dy = \int \frac{1}{x^2(2x+1)}\,dx$$

$$\Rightarrow \ln y = \int\left(\frac{1}{x^2} - \frac{2}{x} + \frac{4}{2x+1}\right)dx \qquad \text{(from part (i))}$$

$$\Rightarrow \ln y = \int x^{-2}\,dx - \int \frac{2}{x}\,dx + 2\int \frac{2}{2x+1}\,dx$$

$$\Rightarrow \ln y = -\frac{1}{x} - 2\ln x + 2\ln|2x+1| + K \;\cdots\cdots\cdots(1)$$

when $x = 1,\; y = 1$

$$\Rightarrow \ln 1 = -\frac{1}{1} - 2\ln 1 + 2\ln|2(1)+1| + K \Rightarrow 0 = -1 + 2\ln(3) + K \Rightarrow K = 1 - 2\ln 3$$

$\therefore$ equation (1) becomes,

$$\ln y = -\frac{1}{x} - 2\ln x + 2\ln|2x+1| + 1 - 2\ln 3$$

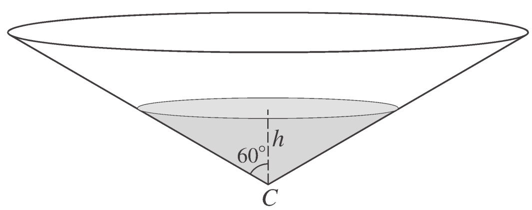
$$\Rightarrow \ln y = -\frac{1}{x} + 1 + 2\left(\ln|2x+1| - \ln x - \ln 3\right) \Rightarrow \ln y = -\frac{1}{x} + 1 + 2\ln\left|\frac{2x+1}{3x}\right|$$

when $x = 2$,

$$\ln y = -\frac{1}{2} + 1 + 2\ln\left|\frac{2(2)+1}{3(2)}\right| \Rightarrow \ln y = \frac{1}{2} + 2\ln\left(\frac{5}{6}\right) \Rightarrow \ln y = \frac{1}{2}\ln e + \ln\left(\frac{5}{6}\right)^2$$

$$\Rightarrow \ln y = \ln e^{\frac{1}{2}} + \ln\frac{25}{36} \Rightarrow \ln y = \ln\frac{25}{36}e^{\frac{1}{2}} \Rightarrow y = \frac{25}{36}e^{\frac{1}{2}} \quad \textbf{(Ans).}$$

Question 9



A tank containing water is in the form of a cone with vertex C. The axis is vertical and the semivertical angle is 60°, as shown in the diagram. At time $t = 0$, the tank is full and the depth of water is H. At this instant, a tap at C is opened and water begins to flow out. The volume of water in the tank decreases at a rate proportional to $\sqrt{h}$, where h is the depth of water at time t. The tank becomes empty when $t = 60$.

(i) Show that h and t satisfy a differential equation of the form

$$\frac{dh}{dt} = -Ah^{-\frac{3}{2}},$$

where A is a positive constant. [4]

(ii) Solve the differential equation given in part **(i)** and obtain an expression for t in terms of h and H. [6]

(iii) Find the time at which the depth reaches $\frac{1}{2}H$. [1]

[The volume V of a cone of vertical height h and base radius r is given by $V = \frac{1}{3}\pi r^2 h$.]

[N13/P3/Q10]

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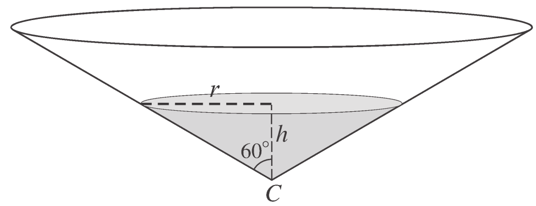
Suggested Solution:

(i) Let r be the radius of the circular surface of water at time t

$\therefore \quad \tan 60° = \dfrac{r}{h} \;\Rightarrow\; r = h\tan 60° \;\Rightarrow\; r = \sqrt{3}\,h$

volume of water at time t is,

$V = \dfrac{1}{3}\pi r^2 h \;\Rightarrow\; V = \dfrac{1}{3}\pi(\sqrt{3}\,h)^2 h \;\Rightarrow\; V = \pi h^3$



differentiating w.r.t. h, $\dfrac{dV}{dh} = 3\pi h^2$

Given that, $\dfrac{dV}{dt} \propto -\sqrt{h} \;\Rightarrow\; \dfrac{dV}{dt} = -k\sqrt{h}$

applying chain rule, $\dfrac{dV}{dt} = \dfrac{dV}{dh} \times \dfrac{dh}{dt}$

$\Rightarrow\; -k\sqrt{h} = 3\pi h^2 \times \dfrac{dh}{dt} \;\Rightarrow\; \dfrac{dh}{dt} = \dfrac{-k\sqrt{h}}{3\pi h^2} = -\dfrac{kh^{\frac{1}{2}-2}}{3\pi} = -(\dfrac{k}{3\pi})h^{-\frac{3}{2}}$

let $A = \dfrac{k}{3\pi}$, $\quad\therefore\quad \dfrac{dh}{dt} = -Ah^{-\frac{3}{2}}$ **(Shown).**

(ii) $\dfrac{dh}{dt} = -Ah^{-\frac{3}{2}} \;\Rightarrow\; \dfrac{1}{h^{-\frac{3}{2}}}\,dh = -A\,dt \;\Rightarrow\; h^{\frac{3}{2}}\,dh = -A\,dt$

integrating both sides, $\displaystyle\int h^{\frac{3}{2}}\,dh = -A\int dt$

$\Rightarrow\; \dfrac{h^{\frac{5}{2}}}{\frac{5}{2}} = -At + C \;\Rightarrow\; \dfrac{2}{5}h^{\frac{5}{2}} = -At + C$ ....(1)

given that, at $t = 0$, depth $h = H$

$\Rightarrow\; \dfrac{2}{5}H^{\frac{5}{2}} = -A(0) + C \;\Rightarrow\; C = \dfrac{2}{5}H^{\frac{5}{2}}$

$\therefore\; \dfrac{2}{5}h^{\frac{5}{2}} = -At + \dfrac{2}{5}H^{\frac{5}{2}}$

also, given that, at $t = 60$, depth $h = 0$

$\Rightarrow\; \dfrac{2}{5}(0)^{\frac{5}{2}} = -A(60) + \dfrac{2}{5}H^{\frac{5}{2}} \;\Rightarrow\; 60A = \dfrac{2}{5}H^{\frac{5}{2}} \;\Rightarrow\; A = \dfrac{1}{150}H^{\frac{5}{2}}$

$\therefore$ equation (1) becomes,

$\dfrac{2}{5}h^{\frac{5}{2}} = -\dfrac{1}{150}H^{\frac{5}{2}}t + \dfrac{2}{5}H^{\frac{5}{2}}$

$\Rightarrow\; \dfrac{2}{5}h^{\frac{5}{2}} = H^{\frac{5}{2}}\left(-\dfrac{t}{150} + \dfrac{2}{5}\right) \;\Rightarrow\; \dfrac{2}{5}h^{\frac{5}{2}} = H^{\frac{5}{2}}\left(\dfrac{-t+60}{150}\right) \;\Rightarrow\; \dfrac{2}{5}h^{\frac{5}{2}} \times \dfrac{150}{H^{\frac{5}{2}}} = 60 - t$

$\Rightarrow\; 60\left(\dfrac{h}{H}\right)^{\frac{5}{2}} = 60 - t \;\Rightarrow\; t = 60 - 60\left(\dfrac{h}{H}\right)^{\frac{5}{2}} \;\Rightarrow\; t = 60\left(1 - \left(\dfrac{h}{H}\right)^{\frac{5}{2}}\right)$

(iii) When, $h = \dfrac{1}{2}H$, $\quad t = 60\left(1 - \left(\dfrac{\frac{1}{2}H}{H}\right)^{\frac{5}{2}}\right) \;\Rightarrow\; t = 60\left(1 - \left(\dfrac{1}{2}\right)^{\frac{5}{2}}\right)$

$\Rightarrow\; t = 60(1 - 0.17678) = 49.39 \approx 49.4$ **(Ans).**

Question 10

The population of a country at time t years is N millions. At any time, N is assumed to increase at a rate proportional to the product of N and $(1 - 0.01N)$. When $t = 0$, $N = 20$ and $\frac{dN}{dt} = 0.32$.

(i) Treating N and t as continuous variables, show that they satisfy the differential equation

$$\frac{dN}{dt} = 0.02N(1 - 0.01N). \qquad [1]$$

(ii) Solve the differential equation, obtaining an expression for t in terms of N. [8]

(iii) Find the time at which the population will be double its value at $t = 0$. [1]

[J14/P3/Q9]

Suggested Solution:

(i) $\frac{dN}{dt} \propto N(1-0.01N) \quad \Rightarrow \quad \frac{dN}{dt} = kN(1-0.01N)$

given that, $N = 20$, $\frac{dN}{dt} = 0.32$

$\Rightarrow \quad 0.32 = k(20)\big(1-0.01(20)\big) \quad \Rightarrow \quad 0.32 = 16k \quad \Rightarrow \quad k = 0.02$

$\therefore \quad \frac{dN}{dt} = 0.02N(1-0.01N)$ **(Shown).**

(ii) $\frac{dN}{dt} = 0.02N(1-0.01N) \quad \Rightarrow \quad \frac{1}{N(1-0.01N)}\, dN = 0.02\, dt$ ....(1)

resolving $\frac{1}{N(1-0.01N)}$ into partial fractions, we have,

$$\frac{1}{N(1-0.01N)} \equiv \frac{A}{N} + \frac{B}{1-0.01N}$$

$\Rightarrow \quad 1 \equiv A(1-0.01N) + BN$

for $N = 0$, $\quad 1 = A(1-0.01(0)) + B(0) \quad \Rightarrow \quad A = 1$

for $N = \frac{1}{0.01} = 100$, $\quad 1 = A(1-0.01(100)) + B(100) \quad \Rightarrow \quad 1 = 100B \quad \Rightarrow \quad B = 0.01$

$$\therefore \quad \frac{1}{N(1-0.01N)} \equiv \frac{1}{N} + \frac{0.01}{1-0.01N}$$

equation (1) can now be written as, $\left(\frac{1}{N} + \frac{0.01}{1-0.01N}\right) dN = 0.02\, dt$

integrating both sides,

$$\int \left(\frac{1}{N} + \frac{0.01}{1-0.01N}\right) dN = \int 0.02\, dt$$

$$\Rightarrow \quad \ln N - \ln(1-0.01N) = 0.02t + C \quad \Rightarrow \quad \ln\left(\frac{N}{1-0.01N}\right) = 0.02t + C$$

$$\Rightarrow \quad \ln\left(\frac{N}{1-\frac{1}{100}N}\right) = 0.02t + C \quad \Rightarrow \quad \ln\left(\frac{100N}{100-N}\right) = 0.02t + C$$

when $t = 0$, $N = 20$, $\quad \Rightarrow \quad \ln\left(\frac{100(20)}{100-20}\right) = 0.02(0) + C \quad \Rightarrow \quad C = \ln 25$

$$\therefore \quad \ln\left(\frac{100N}{100-N}\right) = 0.02t + \ln 25$$

$\Rightarrow \quad \ln\left(\frac{100N}{100-N}\right) - \ln 25 = 0.02t \quad \Rightarrow \quad \ln\left(\frac{100N}{25(100-N)}\right) = \frac{1}{50}t \quad \Rightarrow \quad t = 50\ln\left(\frac{4N}{100-N}\right)$ **(Ans).**

(iii) $t = 50\ln\left(\dfrac{4N}{100-N}\right)$

when $N = 40$, $t = 50\ln\left(\dfrac{4(40)}{100-40}\right) = 50\ln(\frac{8}{3}) = 49$ years. **(Ans).**

Question 11

In a certain country the government charges tax on each litre of petrol sold to motorists. The revenue per year is R million dollars when the rate of tax is x dollars per litre. The variation of R with x is modelled by the differential equation

$$\frac{dR}{dx} = R\left(\frac{1}{x} - 0.57\right),$$

where R and x are taken to be continuous variables. When $x = 0.5$, $R = 16.8$.

(i) Solve the differential equation and obtain an expression for R in terms of x. [6]

(ii) This model predicts that R cannot exceed a certain amount. Find this maximum value of R. [3]

[N14/P3/Q7]

Suggested Solution:

(i) $\dfrac{dR}{dx} = R\left(\dfrac{1}{x} - 0.57\right) \Rightarrow \dfrac{1}{R}\,dR = \left(\dfrac{1}{x} - 0.57\right)dx$

integrating both sides

$$\int \frac{1}{R}\,dR = \int\left(\frac{1}{x} - 0.57\right)dx$$

$\Rightarrow \ln R = \ln x - 0.57x + C$ ....(1)

given that, $x = 0.5$ when $R = 16.8$

$\Rightarrow \ln 16.8 = \ln 0.5 - 0.57(0.5) + C \Rightarrow C = \ln 16.8 - \ln 0.5 + 0.285 \Rightarrow C = 3.8$

$\therefore$ equation (1) becomes,

$\ln R = \ln x - 0.57x + 3.8$

$\Rightarrow \ln R - \ln x = 3.8 - 0.57x$

$\Rightarrow \ln\dfrac{R}{x} = 3.8 - 0.57x \Rightarrow \dfrac{R}{x} = e^{(3.8-0.57x)} \Rightarrow R = xe^{(3.8-0.57x)}$ **(Ans).**

(ii) $\dfrac{dR}{dx} = R\left(\dfrac{1}{x} - 0.57\right)$

When R is maximum, $\dfrac{dR}{dx} = 0$

$\Rightarrow R\left(\dfrac{1}{x} - 0.57\right) = 0$

since $R \neq 0$, $\therefore \dfrac{1}{x} - 0.57 = 0 \Rightarrow \dfrac{1}{x} = 0.57 \Rightarrow x = \dfrac{1}{0.57} = 1.7544$

from part (i), $R = xe^{(3.8-0.57x)}$

when $x = 1.7544$, $R = 1.7544\,e^{3.8-0.57(1.7544)} \Rightarrow R = 1.7544\,e^{2.8} = 28.8505$

$\therefore$ maximum value of $R = 28.8505 \approx 28.9$ million dollars. **(Ans).**

Question 27

The variables x and θ satisfy the differential equation

$$x\sin^2\theta\frac{\mathrm{d}x}{\mathrm{d}\theta}=\tan^2\theta-2\cot\theta,$$

for $0<\theta<\frac{1}{2}\pi$ and $x>0$. It is given that $x=2$ when $\theta=\frac{1}{4}\pi$.

(a) Show that $\frac{\mathrm{d}}{\mathrm{d}\theta}(\cot^2\theta)=-\frac{2\cot\theta}{\sin^2\theta}$.

(You may assume without proof that the derivative of $\cot\theta$ with respect to θ is $-\operatorname{cosec}^2\theta$.) [1]

(b) Solve the differential equation and find the value of x when $\theta=\frac{1}{6}\pi$. [7]

[N22/P3/Q7]

Suggested Solution:

(a) $\frac{\mathrm{d}}{\mathrm{d}\theta}(\cot^2\theta)=2\cot\theta\times(-\operatorname{cosec}^2\theta)$

$=-2\cot\theta\operatorname{cosec}^2\theta=-\frac{2\cot\theta}{\sin^2\theta}$ **(Shown).**

(b) $x\sin^2\theta\frac{\mathrm{d}x}{\mathrm{d}\theta}=\tan^2\theta-2\cot\theta$

$\Rightarrow\ x\,\mathrm{d}x=\frac{\tan^2\theta-2\cot\theta}{\sin^2\theta}\,\mathrm{d}\theta$

$\Rightarrow\ x\,\mathrm{d}x=\left(\frac{\tan^2\theta}{\sin^2\theta}-\frac{2\cot\theta}{\sin^2\theta}\right)\mathrm{d}\theta$

$\Rightarrow\ x\,\mathrm{d}x=\left(\frac{\frac{\sin^2\theta}{\cos^2\theta}}{\sin^2\theta}-\frac{2\cot\theta}{\sin^2\theta}\right)\mathrm{d}\theta$

$\Rightarrow\ x\,\mathrm{d}x=\left(\frac{1}{\cos^2\theta}-\frac{2\cot\theta}{\sin^2\theta}\right)\mathrm{d}\theta$

$\Rightarrow\ x\,\mathrm{d}x=\left(\sec^2\theta-\frac{2\cot\theta}{\sin^2\theta}\right)\mathrm{d}\theta$

Integrating both sides,

$$\int x\,\mathrm{d}x=\int\left[\sec^2\theta+\left(-\frac{2\cot\theta}{\sin^2\theta}\right)\right]\mathrm{d}\theta$$

$\Rightarrow\ \frac{x^2}{2}=\tan\theta+\cot^2\theta+C$ $\quad\left(\text{Using the result of } \textbf{(a)},\ \int-\frac{2\cot\theta}{\sin^2\theta}=\cot^2\theta\right)$

Given that $x=2$, when $\theta=\frac{\pi}{4}$,

$\Rightarrow\ \frac{2^2}{2}=\tan\frac{\pi}{4}+\cot^2\frac{\pi}{4}+C$

$\Rightarrow\ 2=1+\frac{1}{\tan^2\frac{\pi}{4}}+C\ \Rightarrow\ 2=1+\frac{1}{1^2}+C\ \Rightarrow\ 2=2+C\ \Rightarrow\ C=0$

$\therefore$ The equation becomes, $\frac{1}{2}x^2 = \tan\theta + \cot^2\theta$

When $\theta = \frac{\pi}{6}$

$\Rightarrow \frac{1}{2}x^2 = \tan\frac{\pi}{6} + \cot^2\frac{\pi}{6}$

$\Rightarrow \frac{1}{2}x^2 = \tan\frac{\pi}{6} + \frac{1}{\tan^2\frac{\pi}{6}}$

$\Rightarrow \frac{1}{2}x^2 = \frac{1}{\sqrt{3}} + \frac{1}{\left(\frac{1}{\sqrt{3}}\right)^2}$

$\Rightarrow \frac{1}{2}x^2 = \frac{1}{\sqrt{3}} + 3$

$\Rightarrow x^2 = \frac{2}{\sqrt{3}} + 6 \Rightarrow x^2 = 7.1547 \Rightarrow x = 2.6748 \approx 2.67$ **(Ans).**

Question 28

(a) The variables x and y satisfy the differential equation

$$\frac{dy}{dx} = \frac{4+9y^2}{e^{2x+1}}.$$

It is given that $y = 0$ when $x = 1$.

Solve the differential equation, obtaining an expression for y in terms of x. [7]

(b) State what happens to the value of y as x tends to infinity. Give your answer in an exact form. [1]

[J23/P3/Q8]

Suggested Solution:

(a) $\frac{dy}{dx} = \frac{4+9y^2}{e^{2x+1}}$

$\Rightarrow \frac{1}{4+9y^2}\,dy = \frac{1}{e^{2x+1}}\,dx \Rightarrow \frac{1}{4+9y^2}\,dy = e^{-2x-1}\,dx$

Integrating both sides,

$$\int \frac{1}{4+9y^2}\,dy = \int e^{-2x-1}\,dx$$

$$\Rightarrow \int \frac{1}{9(\frac{4}{9}+y^2)}\,dy = \int e^{-2x-1}\,dx$$

$$\Rightarrow \frac{1}{9}\int \frac{1}{(\frac{2}{3})^2+y^2}\,dy = \int e^{-2x-1}\,dx$$

Note:

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + K$$

$$\Rightarrow \frac{1}{9}\left(\frac{1}{\frac{2}{3}}\tan^{-1}\frac{y}{\frac{2}{3}}\right) = \frac{e^{-2x-1}}{-2} + K$$

$$\Rightarrow \frac{1}{9}\left(\frac{3}{2}\tan^{-1}\frac{3y}{2}\right) = -\frac{1}{2}e^{-2x-1} + K \Rightarrow \frac{1}{6}\tan^{-1}\frac{3y}{2} = -\frac{1}{2}e^{-2x-1} + K$$

When $x=1, \quad y=0$

$$\Rightarrow \frac{1}{6}\tan^{-1}\frac{3(0)}{2}=-\frac{1}{2}e^{-2(1)-1}+K \quad \Rightarrow \quad 0=-\frac{1}{2}e^{-3}+K \quad \Rightarrow \quad K=\frac{1}{2}e^{-3}$$

$\therefore$ The equation becomes,

$$\frac{1}{6}\tan^{-1}\frac{3y}{2}=-\frac{1}{2}e^{-2x-1}+\frac{1}{2}e^{-3}$$

$$\Rightarrow \tan^{-1}\frac{3y}{2}=6\left(-\frac{1}{2}e^{-2x-1}+\frac{1}{2}e^{-3}\right)$$

$$\Rightarrow \tan^{-1}\frac{3y}{2}=-3e^{-2x-1}+3e^{-3}$$

$$\Rightarrow \frac{3y}{2}=\tan\left(-3e^{-2x-1}+3e^{-3}\right)$$

$$\Rightarrow y=\frac{2}{3}\tan\left(-3e^{-2x-1}+3e^{-3}\right) \quad \Rightarrow \quad y=\frac{2}{3}\tan\left(\frac{3}{e^3}-\frac{3}{e^{2x+1}}\right) \quad \textbf{(Ans).}$$

(b) $y=\frac{2}{3}\tan\left(\frac{3}{e^3}-\frac{3}{e^{2x+1}}\right)$

As $x\to\infty, \quad \frac{3}{e^{2x+1}}\to 0$

$$\Rightarrow y\to\frac{2}{3}\tan\left(\frac{3}{e^3}\right)$$

$\therefore$ As x tends to infinity, y approaches to, $\frac{2}{3}\tan\left(\frac{3}{e^3}\right)$ **(Ans).**

Question 29

The variables x and y satisfy the differential equation

$$x^2\frac{dy}{dx}+y^2+y=0.$$

It is given that $x=1$ when $y=1$.

(a) Solve the differential equation to obtain an expression for y in terms of x. [8]

(b) State what happens to the value of y when x tends to infinity. Give your answer in an exact form. [1]

[N23/P3/Q11]

Suggested Solution:

(a) $x^2\frac{dy}{dx}+y^2+y=0$

Separating the variables,

$$\Rightarrow x^2\frac{dy}{dx}+y(y+1)=0$$

$$\Rightarrow x^2\frac{dy}{dx}=-y(y+1) \quad \Rightarrow \quad \frac{1}{y(y+1)}\,dy=-\frac{1}{x^2}\,dx \text{ .... (1)}$$

Expressing $\frac{1}{y(y+1)}$ in partial fractions, we have,

$$\frac{1}{y(y+1)} \equiv \frac{A}{y} + \frac{B}{y+1}$$

$\Rightarrow \ 1 \equiv A(y+1) + By$

for $y = 0$, $\quad 1 = A(0+1) + B(0) \quad \Rightarrow \quad A = 1$

for $y = -1$, $\quad 1 = A(-1+1) + B(-1) \quad \Rightarrow \quad B = -1$

$$\therefore \quad \frac{1}{y(y+1)} = \frac{1}{y} - \frac{1}{y+1}$$

Equation (1) can now be written as, $\left(\frac{1}{y} - \frac{1}{y+1}\right) \mathrm{d}y = -\frac{1}{x^2}\ \mathrm{d}x$

Integrating both sides,

$$\int \left(\frac{1}{y} - \frac{1}{y+1}\right) \mathrm{d}y = -\int x^{-2}\ \mathrm{d}x$$

$$\Rightarrow \ \ln y - \ln(y+1) = -\left(\frac{x^{-1}}{-1}\right) + C \quad \Rightarrow \quad \ln\left(\frac{y}{y+1}\right) = \frac{1}{x} + C \ .... (2)$$

Given that, $x = 1$ when $y = 1$,

$$\Rightarrow \ \ln\left(\frac{1}{1+1}\right) = \frac{1}{1} + C \ \Rightarrow \ \ln\left(\frac{1}{2}\right) = 1 + C \ \Rightarrow \ C = \ln 1 - \ln 2 - 1 \ \Rightarrow \ C = -\ln 2 - 1$$

$\therefore$ Equation (2) becomes, $\quad \ln\left(\frac{y}{y+1}\right) = \frac{1}{x} - \ln 2 - 1$

$$\Rightarrow \ \ln\left(\frac{y}{y+1}\right) + \ln 2 = \frac{1}{x} - 1$$

$$\Rightarrow \ \ln\left(\frac{2y}{y+1}\right) = \frac{1}{x} - 1$$

$$\Rightarrow \ \frac{2y}{y+1} = \mathrm{e}^{\frac{1}{x}-1}$$

$$\Rightarrow \ 2y = \mathrm{e}^{\frac{1}{x}-1}(y+1)$$

$$\Rightarrow \ 2y = y\mathrm{e}^{\frac{1}{x}-1} + \mathrm{e}^{\frac{1}{x}-1}$$

$$\Rightarrow \ 2y - y\mathrm{e}^{\frac{1}{x}-1} = \mathrm{e}^{\frac{1}{x}-1} \quad \Rightarrow \quad y\left(2 - \mathrm{e}^{\frac{1}{x}-1}\right) = \mathrm{e}^{\frac{1}{x}-1} \quad \Rightarrow \quad y = \frac{\mathrm{e}^{\frac{1}{x}-1}}{2 - \mathrm{e}^{\frac{1}{x}-1}} \quad \textbf{(Ans).}$$

Remember:

- $\ln(a \times b) = \ln a + \ln b$
- $\ln(\frac{a}{b}) = \ln a - \ln b$
- $\ln(a^b) = b \ln a$
- $\ln \mathrm{e} = 1$
- $\ln 1 = 0$

(b) $y = \frac{\mathrm{e}^{\frac{1}{x}-1}}{2 - \mathrm{e}^{\frac{1}{x}-1}}$. When $x \to \infty$, then, $\frac{1}{x} \to 0$

$$\Rightarrow \ y \to \frac{\mathrm{e}^{0-1}}{2 - \mathrm{e}^{0-1}}$$

$$= \frac{\mathrm{e}^{-1}}{2 - \mathrm{e}^{-1}}$$

$$= \frac{\frac{1}{\mathrm{e}}}{2 - \frac{1}{\mathrm{e}}} = \frac{\frac{1}{\mathrm{e}}}{\frac{2\mathrm{e}-1}{\mathrm{e}}} = \frac{1}{\mathrm{e}} \times \frac{\mathrm{e}}{2\mathrm{e}-1} = \frac{1}{2\mathrm{e}-1}$$

$\therefore$ When x tends to infinity, y approaches to, $\frac{1}{2\mathrm{e}-1}$ **(Ans).**

Question 30

(a) By writing $y = \sec^3\theta$ as $\dfrac{1}{\cos^3\theta}$, show that $\dfrac{dy}{d\theta} = 3\sin\theta\sec^4\theta$. [2]

(b) The variables x and θ satisfy the differential equation

$$(x^2+9)\sin\theta\frac{d\theta}{dx} = (x+3)\cos^4\theta.$$

It is given that $x = 3$ when $\theta = \frac{1}{3}\pi$.

Solve the differential equation to find the value of $\cos\theta$ when $x = 0$. Give your answer correct to 3 significant figures. [8]

[J24/P3/Q10]

Suggested Solution:

(a) $y = \sec^3\theta$

$\Rightarrow \quad y = \dfrac{1}{\cos^3\theta} \quad \Rightarrow \quad y = (\cos\theta)^{-3}$

Differentiating w.r.t. θ,

$$\frac{d}{d\theta}(y) = \frac{d}{d\theta}(\cos\theta)^{-3}$$

$$\Rightarrow \quad \frac{dy}{d\theta} = -3(\cos\theta)^{-4}\times\frac{d}{d\theta}(\cos\theta)$$

$$\Rightarrow \quad \frac{dy}{d\theta} = -\frac{3}{\cos^4\theta}\times(-\sin\theta)$$

$$\Rightarrow \quad \frac{dy}{d\theta} = \frac{3\sin\theta}{\cos^4\theta} \quad \Rightarrow \quad \frac{dy}{d\theta} = 3\sin\theta\sec^4\theta \quad \textbf{(Shown).}$$

(b) $(x^2+9)\sin\theta\,\dfrac{d\theta}{dx} = (x+3)\cos^4\theta$

Separating the variables, we have,

$$\frac{\sin\theta}{\cos^4\theta}\,d\theta = \frac{x+3}{x^2+9}\,dx \quad \Rightarrow \quad \sin\theta\sec^4\theta\,d\theta = \frac{x+3}{x^2+9}\,dx$$

Integrating both sides

$$\int \sin\theta\sec^4\theta\,d\theta = \int\frac{x+3}{x^2+9}\,dx$$

$$\Rightarrow \quad \int \sin\theta\sec^4\theta\,d\theta = \int\frac{x}{x^2+9}\,dx + \int\frac{3}{x^2+9}\,dx$$

From part (**a**), we see that, $\dfrac{d}{d\theta}(\sec^3\theta) = 3\sin\theta\sec^4\theta$

$$\therefore \quad \int 3\sin\theta\sec^4\theta\,d\theta = \sec^3\theta + C$$

$$\Rightarrow \quad \int \sin\theta\sec^4\theta\,d\theta = \frac{1}{3}\sec^3\theta + C$$

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Using this result, we have,

$$\frac{1}{3}\sec^3\theta = \int \frac{x}{x^2+9}\,\mathrm{d}x + \int \frac{3}{x^2+9}\,\mathrm{d}x$$

$$\Rightarrow \frac{1}{3}\sec^3\theta = \frac{1}{2}\int \frac{2x}{x^2+9}\,\mathrm{d}x + 3\int \frac{1}{x^2+3^2}\,\mathrm{d}x$$

$$\Rightarrow \frac{1}{3}\sec^3\theta = \frac{1}{2}\ln(x^2+9) + 3\left(\frac{1}{3}\tan^{-1}\frac{x}{3}\right) + K$$

$$\Rightarrow \frac{1}{3}\sec^3\theta = \frac{1}{2}\ln(x^2+9) + \tan^{-1}\frac{x}{3} + K \text{ .... (1)}$$

Note:

$$\int \frac{1}{a^2+x^2}\,\mathrm{d}x = \frac{1}{a}\tan^{-1}\frac{x}{a} + K$$

Given that, $x = 3$, when $\theta = \frac{\pi}{3}$

$$\Rightarrow \frac{1}{3}\left(\sec\frac{\pi}{3}\right)^3 = \frac{1}{2}\ln(3^2+9) + \tan^{-1}\frac{3}{3} + K$$

$$\Rightarrow \frac{1}{3}\left(\frac{1}{\cos\frac{\pi}{3}}\right)^3 = \frac{1}{2}\ln(18) + \tan^{-1}(1) + K$$

$$\Rightarrow \frac{1}{3}\left(\frac{1}{\frac{1}{2}}\right)^3 = \frac{1}{2}\ln(18) + \frac{\pi}{4} + K$$

$$\Rightarrow \frac{1}{3}(2)^3 = \frac{1}{2}\ln(18) + \frac{\pi}{4} + K \quad \Rightarrow \quad K = \frac{8}{3} - \frac{1}{2}\ln(18) - \frac{\pi}{4}$$

$\therefore$ the equation becomes

$$\frac{1}{3}\sec^3\theta = \frac{1}{2}\ln(x^2+9) + \tan^{-1}\frac{x}{3} + \frac{8}{3} - \frac{1}{2}\ln(18) - \frac{\pi}{4}$$

When $x = 0$,

$$\frac{1}{3}\sec^3\theta = \frac{1}{2}\ln(0^2+9) + \tan^{-1}\frac{0}{3} + \frac{8}{3} - \frac{1}{2}\ln(18) - \frac{\pi}{4}$$

$$\Rightarrow \frac{1}{3}\left(\frac{1}{\cos^3\theta}\right) = \frac{1}{2}\ln 9 + \frac{8}{3} - \frac{1}{2}\ln(18) - \frac{\pi}{4}$$

$$\Rightarrow \frac{1}{3}\left(\frac{1}{\cos^3\theta}\right) = 1.0986 + 2.6667 - 1.4452 - 0.7854$$

$$\Rightarrow \frac{1}{\cos^3\theta} = 3(1.5347)$$

$$\Rightarrow \frac{1}{\cos^3\theta} = 4.6041$$

$$\Rightarrow \cos^3\theta = \frac{1}{4.6041}$$

$$\Rightarrow \cos\theta = \sqrt[3]{\frac{1}{4.6041}} \quad \Rightarrow \quad \cos\theta = 0.601 \quad \textbf{(Ans).}$$

Question 31

A balloon in the shape of a sphere has volume V and radius r. Air is pumped into the balloon at a constant rate of 40π starting when time $t = 0$ and $r = 0$. At the same time, air begins to flow out of the balloon at a rate of $0.8\pi r$. The balloon remains a sphere at all times.

(a) Show that r and t satisfy the differential equation

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \frac{50 - r}{5r^2}. \qquad [3]$$

(b) Find the quotient and remainder when $5r^2$ is divided by $50 - r$. [3]

(c) Solve the differential equation in part **(a)**, obtaining an expression for t in terms of r. [6]

(d) Find the value of t when the radius of the balloon is 12. [1]

[N24/P3/Q10]

Suggested Solution:

(a) Given that, rate of inflow of air, $\frac{\mathrm{d}V}{\mathrm{d}t} = 40\pi$

rate of outflow of air, $\frac{\mathrm{d}V}{\mathrm{d}t} = -0.8\pi r$

$\therefore \quad \frac{\mathrm{d}V}{\mathrm{d}t} = 40\pi - 0.8\pi r$

Volume of sphere, $V = \frac{4}{3}\pi r^3$

Differentiating w.r.t. r, $\frac{\mathrm{d}V}{\mathrm{d}r} = \frac{4}{3}\pi(3r^2) \;\Rightarrow\; \frac{\mathrm{d}V}{\mathrm{d}r} = 4\pi r^2$

Applying chain rule, $\frac{\mathrm{d}r}{\mathrm{d}t} = \frac{\mathrm{d}r}{\mathrm{d}V} \times \frac{\mathrm{d}V}{\mathrm{d}t}$

$\Rightarrow \quad \frac{\mathrm{d}r}{\mathrm{d}t} = \frac{1}{4\pi r^2} \times (40\pi - 0.8\pi r)$

$\Rightarrow \quad \frac{\mathrm{d}r}{\mathrm{d}t} = \frac{1}{4\pi r^2} \times \left(40\pi - \frac{4}{5}\pi r\right)$

$\Rightarrow \quad \frac{\mathrm{d}r}{\mathrm{d}t} = \frac{1}{4\pi r^2} \times 4\pi\left(10 - \frac{r}{5}\right)$

$\Rightarrow \quad \frac{\mathrm{d}r}{\mathrm{d}t} = \frac{1}{r^2} \times \left(\frac{50 - r}{5}\right) \;\Rightarrow\; \frac{\mathrm{d}r}{\mathrm{d}t} = \frac{50 - r}{5r^2}$ **(Shown).**

(b)

$$\begin{array}{r|l} & -5r - 250 \\ \hline 50 - r & 5r^2 \\ & 5r^2 - 250r \\ & \underline{-\qquad +\qquad} \\ & 250r \\ & 250r - 12500 \\ & \underline{-\qquad +\qquad} \\ & 12500 \end{array}$$

$\therefore \quad$ Quotient $= -5r - 250$, Remainder $= 12500$ **(Ans).**

Learning CORNER

(c) $\dfrac{\mathrm{d}r}{\mathrm{d}t} = \dfrac{50-r}{5r^2} \quad \Rightarrow \quad \dfrac{5r^2}{50-r}\ \mathrm{d}r = \mathrm{d}t$

Using the result of part **(b)**,

$$\Rightarrow \left(-5r - 250 + \frac{12500}{50-r}\right)\ \mathrm{d}r = \mathrm{d}t$$

Integrating both sides,

$$\Rightarrow \int \left(-5r - 250 + \frac{12500}{50-r}\right)\ \mathrm{d}r = \int 1\ \mathrm{d}t$$

$$\Rightarrow \int \left(-5r - 250 - 12500\left(\frac{-1}{50-r}\right)\right)\ \mathrm{d}r = \int 1\ \mathrm{d}t$$

$$\Rightarrow -5\left(\frac{r^2}{2}\right) - 250r - 12500\ln(50-r) = t + C$$

Given that, when $t = 0$, $r = 0$,

$$\Rightarrow -5\left(\frac{0^2}{2}\right) - 250(0) - 12500\ln(50-0) = 0 + C$$

$$\Rightarrow 0 - 0 - 12500\ln 50 = C \quad \Rightarrow \quad C = -12500\ln 50$$

$\therefore$ The equation becomes,

$$-\frac{5}{2}r^2 - 250r - 12500\ln(50-r) = t - 12500\ln 50$$

$$\Rightarrow -\frac{5}{2}r^2 - 250r - 12500\ln(50-r) + 12500\ln 50 = t$$

$$\Rightarrow t = -\frac{5}{2}r^2 - 250r + 12500\left(\ln 50 - \ln(50-r)\right)$$

$$\Rightarrow t = -\frac{5}{2}r^2 - 250r + 12500\ln\left(\frac{50}{50-r}\right) \quad \textbf{(Ans).}$$

(d) From part **(c)**, $t = -\dfrac{5}{2}r^2 - 250r + 12500\ln\left(\dfrac{50}{50-r}\right)$

when, $r = 12$,

$$t = -\frac{5}{2}(12)^2 - 250(12) + 12500\ln\left(\frac{50}{50-12}\right)$$

$$\Rightarrow t = -360 - 3000 + 12500\ln\left(\frac{50}{38}\right)$$

$$\Rightarrow t = -3360 + 12500\ln\left(\frac{25}{19}\right)$$

$$\Rightarrow t = -3360 + 3430.46 \quad \Rightarrow \quad t = 70.46 \approx 70.5 \quad \textbf{(Ans).}$$