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About AS LEVEL

MATHEMATICS
Paper 1
(TOPICAL)

period	2010 to 2025
contents	June & November, P1, Worked Solutions
form	Topic By Topic
compiled for	AS Levels
special features	Learning Corner

About **Learning Corner**

This column will further enrich students with the extra information and knowledge provided. It improves the students' ability to solve problems of similar style. Other important guidances are also stated.

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Topic 2
Functions and Graphs

Question 1

The functions f and g are defined for $x \in \mathbb{R}$ by

$$f : x \mapsto 4x - 2x^2,$$
$$g : x \mapsto 5x + 3.$$

(i) Find the range of f. [2]

(ii) Find the value of the constant k for which the equation $gf(x) = k$ has equal roots. [3]

[J10/P1/Q3]

Suggested Solution:

(i) $f(x) = 4x - 2x^2$, for $x \in \mathbb{R}$

$\Rightarrow \quad f(x) = 2x(2 - x)$

Critical values are $x = 0$ and $x = 2$ and the parabolic curve open downwards.

$\therefore \quad$ maximum value of $f(x)$ is at $x = 1$, which is

$f(1) = 4(1) - 2(1)^2 \quad \Rightarrow \quad f(1) = 2$

$\therefore \quad$ range of $f(x)$ is: $\quad f(x) \le 2 \quad$ **(Ans).**

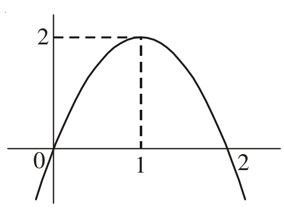
Alternative Solution:

Using completing the square method,

$$\begin{aligned} f(x) &= 4x - 2x^2 \\ &= -2(x^2 - 2x) \\ &= -2(x^2 - 2x + 1 - 1) \\ &= -2\left((x-1)^2 - 1\right) = -2(x-1)^2 + 2 \end{aligned}$$

$\Rightarrow \quad$ max. point has coordinates (1, 2)

$\therefore \quad$ range of $f(x)$ is: $\quad f(x) \le 2 \quad$ **(Ans).**



(ii) $gf(x) = g(4x - 2x^2)$

$$= 5(4x - 2x^2) + 3 = 20x - 10x^2 + 3$$

given that $gf(x) = k$, has equal roots,

$\Rightarrow \quad 20x - 10x^2 + 3 = k \quad \Rightarrow \quad 10x^2 - 20x + (k - 3) = 0$

Discriminant, $\quad b^2 - 4ac = 0$

$\Rightarrow \quad (-20)^2 - 4(10)(k - 3) = 0$

$\Rightarrow \quad 400 - 40(k - 3) = 0$

$\Rightarrow \quad 40(k - 3) = 400$

$\Rightarrow \quad k - 3 = 10 \quad \Rightarrow \quad k = 13 \quad$ **(Ans).**

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Question 2

The function f is defined by

$$f(x) = x^2 - 4x + 7 \quad \text{for } x > 2.$$

(i) Express f(x) in the form $(x-a)^2 + b$ and hence state the range of f. [3]

(ii) Obtain an expression for $f^{-1}(x)$ and state the domain of f^{-1}. [3]

The function g is defined by

$$g(x) = x - 2 \quad \text{for } x > 2.$$

The function h is such that f = hg and the domain of h is $x > 0$.

(iii) Obtain an expression for h(x). [1]

[N10/P1/Q7]

Suggested Solution:

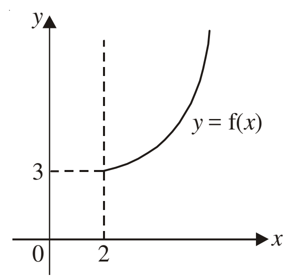
(i) $f(x) = x^2 - 4x + 7$

$= x^2 - 2(x)(2) + (2)^2 - (2)^2 + 7$

$= (x-2)^2 - 4 + 7 = (x-2)^2 + 3$ **(Ans).**

we see that the turning point is (2, 3) and the curve open upwards,

$\therefore$ range of f is: $f(x) \geq 3$ **(Ans).**



Note that f(x) is valid for $x > 2$.
So f(x) is on the right side of the axis of symmetry.
Therefore only keep the positive sign with the radical.

(ii) From part **(i)**, $f(x) = (x-2)^2 + 3$

let $y = f(x), \Rightarrow f^{-1}(y) = x$

$\therefore \quad y = (x-2)^2 + 3 \Rightarrow (x-2)^2 = y - 3 \Rightarrow x - 2 = \pm\sqrt{y-3}$

The graph of f(x) is to the right of the axis of symmetry, therefore using positive sign with the radical, we have

$x - 2 = \sqrt{y-3} \Rightarrow x = 2 + \sqrt{y-3}$

or $f^{-1}(y) = 2 + \sqrt{y-3} \Rightarrow f^{-1}(x) = 2 + \sqrt{x-3}$ **(Ans).**

Now, $f(x) = (x-2)^2 + 3$

range of f is: $f(x) \geq 3$

$\Rightarrow$ domain of $f^{-1}(x)$ is: $x \geq 3$ **(Ans).**

Domain of $f^{-1}(x)$ is the range of $f(x)$ and vice versa.

(iii) $f(x) = (x-2)^2 + 3, \quad g(x) = x - 2$

Given that, f = hg

$\Rightarrow h = f g^{-1}$

$\Rightarrow h(x) = f(x+2) \quad$ (since $g^{-1}(x) = x + 2$)

$\Rightarrow h(x) = [(x+2) - 2]^2 + 3$

$\Rightarrow h(x) = x^2 + 3$

Question 3

The function f is defined by $f : x \mapsto \dfrac{x+3}{2x-1}, x \in \mathbb{R}, x \neq \dfrac{1}{2}$.

(i) Show that ff(x) = x. [3]

(ii) Hence, or otherwise, obtain an expression for $f^{-1}(x)$. [2]

[J11/P1/Q6]

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Suggested Solution:

(i) $\mathrm{f}(x)=\dfrac{x+3}{2x-1}$

$$\mathrm{ff}(x)=\frac{\mathrm{f}(x)+3}{2\mathrm{f}(x)-1}$$

$$=\frac{\frac{x+3}{2x-1}+3}{2\left(\frac{x+3}{2x-1}\right)-1}$$

$$=\frac{\frac{x+3+3(2x-1)}{2x-1}}{\frac{2x+6-(2x-1)}{2x-1}}=\frac{x+3+6x-3}{2x+6-2x+1}=\frac{7x}{7}=x \quad \textbf{(Shown).}$$

(ii) $\mathrm{f}(x)=\dfrac{x+3}{2x-1}$

Let $\mathrm{f}(x)=y$

$\Rightarrow \quad y=\dfrac{x+3}{2x-1}$

$2xy-y=x+3$

$2xy-x=y+3$

$x(2y-1)=y+3$

$x=\dfrac{y+3}{2y-1}$

as $\mathrm{f}(x)=y \quad \Rightarrow \quad x=\mathrm{f}^{-1}(y)$

$\therefore \quad \mathrm{f}^{-1}(y)=\dfrac{y+3}{2y-1} \quad \Rightarrow \quad \mathrm{f}^{-1}(x)=\dfrac{x+3}{2x-1}$ **(Ans).**

Alternative Solution to part (ii):

From part **(i)**, $\mathrm{ff}(x)=x$

$\Rightarrow$ ff is an identity function

$\Rightarrow \quad \mathrm{ff}=1$

$\Rightarrow \quad \mathrm{f}=\mathrm{f}^{-1}$

$\therefore \quad \mathrm{f}^{-1}(x)=\dfrac{x+3}{2x-1}$ **(Ans).**

Question 4

The functions f and g are defined for $x\in\mathbb{R}$ by

$$\mathrm{f}:x\mapsto 3x+a,$$
$$\mathrm{g}:x\mapsto b-2x,$$

where a and b are constants. Given that $\mathrm{ff}(2)=10$ and $g^{-1}(2)=3$, find

(i) the values of a and b, [4]

(ii) an expression for $\mathrm{fg}(x)$. [2]

[N11/P1/Q2]

Suggested Solution:

(i) $\mathrm{f}(x)=3x+a$

$\mathrm{f}(2)=3(2)+a=6+a$

$\therefore \quad \mathrm{ff}(2)=\mathrm{f}(6+a)=3(6+a)+a=18+4a$

given that, $\mathrm{ff}(2)=10$

$\Rightarrow \quad 18+4a=10 \quad \Rightarrow \quad a=-2$ **(Ans).**

$\mathrm{g}(x)=b-2x$

given that, $\mathrm{g}^{-1}(2)=3$

$\Rightarrow \quad \mathrm{g}(3)=2$

$\Rightarrow \quad b-2(3)=2$

$\Rightarrow \quad b-6=2 \quad \Rightarrow \quad b=8$ **(Ans).**

(ii) Using the results of part **(i)**,

$\mathrm{f}(x)=3x-2$, and $\mathrm{g}(x)=8-2x$

$\therefore \quad \mathrm{fg}(x)=\mathrm{f}(8-2x)$

$=3(8-2x)-2$

$=24-6x-2=22-6x$ **(Ans).**

Question 5

Functions f and g are defined by

$$\mathrm{f} : x \mapsto 2x + 5 \quad \text{for } x \in \mathbb{R},$$

$$\mathrm{g} : x \mapsto \frac{8}{x-3} \quad \text{for } x \in \mathbb{R},\ x \neq 3.$$

(i) Obtain expressions, in terms of x, for $\mathrm{f}^{-1}(x)$ and $\mathrm{g}^{-1}(x)$, stating the value of x for which $\mathrm{g}^{-1}(x)$ is not defined. [4]

(ii) Sketch the graphs of $y = \mathrm{f}(x)$ and $y = \mathrm{f}^{-1}(x)$ on the same diagram, making clear the relationship between the two graphs. [3]

(iii) Given that the equation $\mathrm{fg}(x) = 5 - kx$, where k is a constant, has no solutions, find the set of possible values of k. [5]

[J12/P1/Q10]

Suggested Solution:

(i) $\mathrm{f}(x) = 2x + 5 \qquad x \in \mathbb{R}$

Let $y = \mathrm{f}(x)$

$\Rightarrow \quad y = 2x + 5$

$\Rightarrow \quad 2x = y - 5 \quad \Rightarrow \quad x = \dfrac{y-5}{2}$

since, $\quad \mathrm{f}(x) = y \quad \Rightarrow \quad x = \mathrm{f}^{-1}(y)$

$\therefore \quad \mathrm{f}^{-1}(y) = \dfrac{y-5}{2}$

$\Rightarrow \quad \mathrm{f}^{-1}(x) = \dfrac{x-5}{2}$ **(Ans).**

$\mathrm{g}(x) = \dfrac{8}{x-3}, \qquad x \in \mathbb{R}, \quad x \neq 3$

Let $\mathrm{g}(x) = z$

$\Rightarrow \quad z = \dfrac{8}{x-3}$

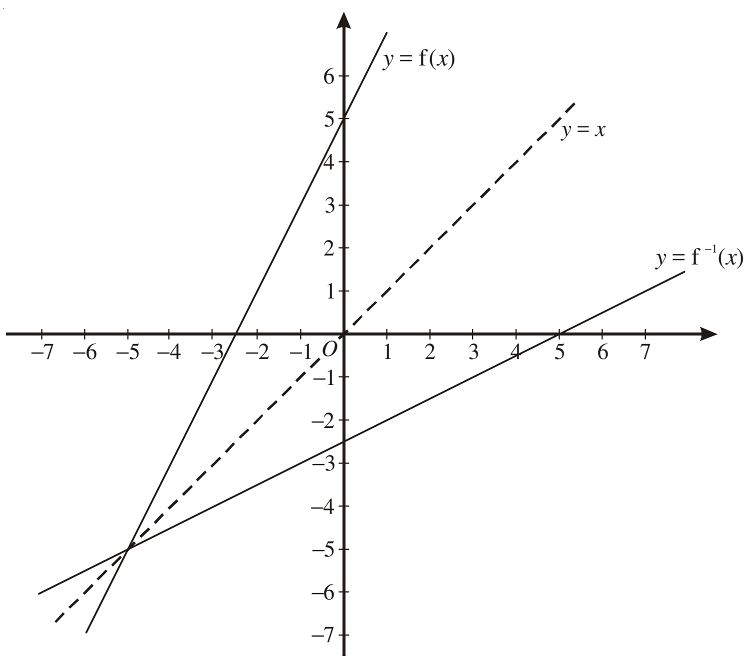
$\Rightarrow \quad x - 3 = \dfrac{8}{\mathrm{z}} \quad \Rightarrow \quad x = \dfrac{8}{\mathrm{z}} + 3$

since, $\quad \mathrm{g}(x) = z \quad \Rightarrow \quad x = \mathrm{g}^{-1}(z)$

$\therefore \quad \mathrm{g}^{-1}(z) = \dfrac{8}{z} + 3$

$\Rightarrow \quad \mathrm{g}^{-1}(x) = \dfrac{8}{x} + 3, \quad x \in \mathbb{R}, \ x \neq 0$ **(Ans).**

(ii)



The graph of $y = \mathrm{f}^{-1}(x)$ is a reflection of graph of $y = \mathrm{f}(x)$ in the line $y = x$.

(iii) $\mathrm{fg}(x) = 5 - kx$

$\Rightarrow \quad \mathrm{f}\left(\dfrac{8}{x-3}\right) = 5 - kx$

$\Rightarrow \quad 2\left(\dfrac{8}{x-3}\right) + 5 = 5 - kx$

$\Rightarrow \quad \dfrac{16}{x-3} = -kx \quad \Rightarrow \quad 16 = -kx(x-3) \quad \Rightarrow \quad kx^2 - 3kx + 16 = 0$

as the equation has no solution,

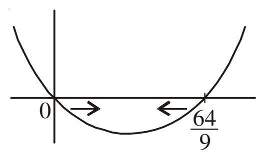
$\therefore$ discriminant, $b^2 - 4ac < 0$

$\Rightarrow \quad (-3k)^2 - 4(k)(16) < 0$

$9k^2 - 64k < 0$

$k(9k - 64) < 0$

critical values are, $k = 0$ and $k = \dfrac{64}{9}$



$\therefore \quad 0 < k < \dfrac{64}{9}$ **(Ans).**

Question 6

A function f is such that $\mathrm{f}(x) = \sqrt{\left(\dfrac{x+3}{2}\right)} + 1$, for $x \geq -3$. Find

(i) $\mathrm{f}^{-1}(x)$ in the form $ax^2 + bx + c$, where a, b and c are constants, [3]

(ii) the domain of f^{-1}. [1]

[N12/P1/Q2]

Suggested Solution:

(i) $\mathrm{f}(x) = \sqrt{\left(\dfrac{x+3}{2}\right)} + 1$

Let $y = \mathrm{f}(x)$

$\Rightarrow \quad y = \sqrt{\left(\dfrac{x+3}{2}\right)} + 1 \quad \Rightarrow \quad y - 1 = \sqrt{\dfrac{x+3}{2}}$

squaring both sides,

$\Rightarrow \quad (y-1)^2 = \dfrac{x+3}{2}$

$\Rightarrow \quad x + 3 = 2(y-1)^2$

$\Rightarrow \quad x = 2(y-1)^2 - 3$

since, $\mathrm{f}(x) = y \quad \Rightarrow \quad x = \mathrm{f}^{-1}(y)$

$\therefore \quad \mathrm{f}^{-1}(y) = 2(y-1)^2 - 3$

$\Rightarrow \quad \mathrm{f}^{-1}(x) = 2(x-1)^2 - 3$

$= 2(x^2 - 2x + 1) - 3 = 2x^2 - 4x - 1$ **(Ans).**

(ii) $\mathrm{f}(x) = \sqrt{\dfrac{x+3}{2}} + 1$, for $x \geq -3$

$\therefore$ range of $\mathrm{f}(x)$ is: $\mathrm{f}(x) \geq 1$

$\Rightarrow$ domain of $\mathrm{f}^{-1}(x)$ is: $x \geq 1$ **(Ans).**

Domain of $\mathrm{f}^{-1}(x)$ is the range of $\mathrm{f}(x)$ and vice versa.

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Question 7

A function f is defined by $f(x) = \dfrac{5}{1-3x}$, for $x \geq 1$.

(i) Find an expression for $f'(x)$. [2]

(ii) Determine, with a reason, whether f is an increasing function, a decreasing function or neither. [1]

(iii) Find an expression for $f^{-1}(x)$, and state the domain and range of f^{-1}. [5]

[J13/P1/Q9]

Suggested Solution:

(i) $f(x) = \dfrac{5}{1-3x} = 5(1-3x)^{-1}$

Differentiating w.r.t. x

$$f'(x) = -5(1-3x)^{-2} \times -3$$

$$= \frac{15}{(1-3x)^2} \quad \textbf{(Ans).}$$

Note that: $f'(x) = \dfrac{dy}{dx}$

(ii) $f'(x) = \dfrac{15}{(1-3x)^2}$

$(1-3x)^2$ is a squared quantity and is always positive for $x \geq 1$

$\Rightarrow \quad f'(x) = \dfrac{15}{(1-3x)^2} > 0$

$\therefore \quad f(x)$ is an increasing function. **(Ans).**

Note that:

- $f(x)$ is an increasing function if $f'(x) > 0$
- $f(x)$ is a decreasing function if $f'(x) < 0$.

(iii) $f(x) = \dfrac{5}{1-3x}$

Let $f(x) = y$

$\Rightarrow \quad y = \dfrac{5}{1-3x}$

$\Rightarrow \quad 1-3x = \dfrac{5}{y} \quad \Rightarrow \quad 3x = 1 - \dfrac{5}{y} \quad \Rightarrow \quad x = \dfrac{1}{3}\left(1 - \dfrac{5}{y}\right) \quad \Rightarrow \quad x = \dfrac{y-5}{3y}$

$\Rightarrow \quad f^{-1}(y) = \dfrac{y-5}{3y} \qquad (\because \ x = f^{-1}(y))$

$\therefore \quad f^{-1}(x) = \dfrac{x-5}{3x} \quad$ **(Ans).**

now, $f(x) = \dfrac{5}{1-3x}$, for $x \geq 1$

for $x = 1$, $f(1) = \dfrac{5}{1-3(1)} = -\dfrac{5}{2} = -2.5$

also, we observe that as $x \to \infty$, $f(x) \to 0$

$\therefore \quad$ range of $f(x)$ is: $-2.5 \leq f(x) < 0$

$\Rightarrow \quad$ domain of $f^{-1}(x)$ is: $-2.5 \leq x < 0$ **(Ans).**

domain of $f(x)$ is: $x \geq 1$

$\Rightarrow \quad$ range of $f^{-1}(x)$ is: $f^{-1}(x) \geq 1$ **(Ans).**

Domain of $f^{-1}(x)$ is the range of $f(x)$ and vice versa.

Question 32

The graph of $y = f(x)$ is transformed to the graph of $y = f(2x) - 3$.

(a) Describe fully the two single transformations that have been combined to give the resulting transformation. [3]

The point $P(5,\ 6)$ lies on the transformed curve $y = f(2x) - 3$.

(b) State the coordinates of the corresponding point on the original curve $y = f(x)$. [2]

[N21/P1/Q2]

Suggested Solution:

(a) 1. Horizontal stretch by scale factor $\frac{1}{2}$ along positive x-axis.

2. Translation of 3 units along negative y-axis, i.e. translation of $\begin{pmatrix} 0 \\ -3 \end{pmatrix}$.

(b) Let $(x,\ y)$ be the corresponding point on curve $y = f(x)$.

$(x,\ y)$ is stretched by factor $\frac{1}{2}$ along positive x-axis and then given a translation of $\begin{pmatrix} 0 \\ -3 \end{pmatrix}$.

Therefore $(x,\ y)$ changes to $\left(\frac{1}{2}x,\ \ y-3\right)$

$\therefore\ \left(\frac{1}{2}x,\ \ y-3\right) = (5,\ 6)$

$\Rightarrow\ \frac{1}{2}x = 5 \quad \Rightarrow \quad x = 10$

$y - 3 = 6 \quad \Rightarrow \quad y = 9$

$\therefore$ corresponding point on curve $y = f(x)$ is $(10,\ 9)$ **(Ans).**

Question 33

The function f is defined as follows:

$$f(x) = \frac{x+3}{x-1} \quad \text{for } x > 1.$$

(a) Find the value of ff(5). [2]

(b) Find an expression for $f^{-1}(x)$. [3]

[N21/P1/Q3]

Suggested Solution:

(a) $f(x) = \dfrac{x+3}{x-1}$

$f(5) = \dfrac{5+3}{5-1} = \dfrac{8}{4} = 2$

Now, $ff(5) = f(2)$

$= \dfrac{2+3}{2-1} = 5$ **(Ans).**

(b) $f(x)=\dfrac{x+3}{x-1}$

Let $y=f(x)$

$\Rightarrow\ y=\dfrac{x+3}{x-1}$

$\Rightarrow\ xy-y=x+3$

$\Rightarrow\ xy-x=y+3\ \Rightarrow\ x(y-1)=y+3\ \Rightarrow\ x=\dfrac{y+3}{y-1}$

Since, $f(x)=y\ \Rightarrow\ x=f^{-1}(y)$

$\Rightarrow\ f^{-1}(y)=\dfrac{y+3}{y-1}$

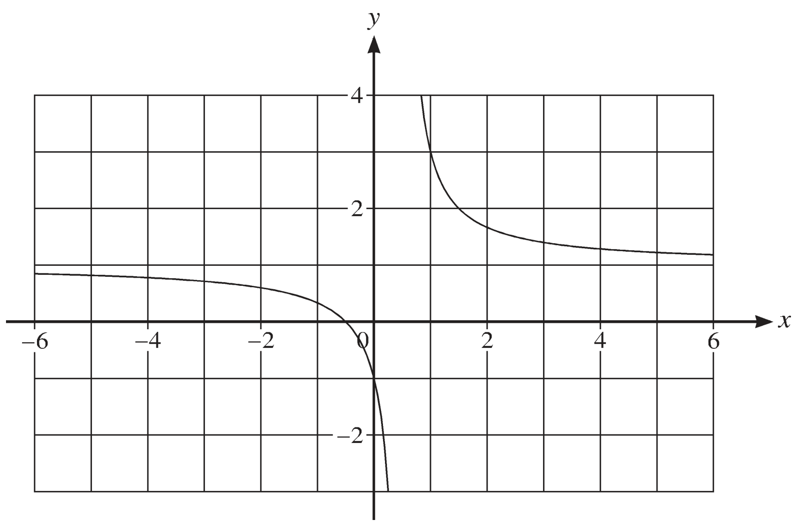
$\therefore\ f^{-1}(x)=\dfrac{x+3}{x-1}$ **(Ans).**

Question 34

Functions f and g are defined as follows:

$$f(x)=\frac{2x+1}{2x-1}\quad \text{for } x\neq\frac{1}{2},$$
$$g(x)=x^2+4\quad \text{for } x\in\mathbb{R}.$$

(a)



The diagram shows part of the graph of $y=f(x)$.

State the domain of f^{-1}. [1]

(b) Find an expression for $f^{-1}(x)$. [3]

(c) Find $gf^{-1}(3)$. [2]

(d) Explain why $g^{-1}(x)$ cannot be found. [1]

(e) Show that $1+\dfrac{2}{2x-1}$ can be expressed as $\dfrac{2x+1}{2x-1}$. Hence find the area of the triangle enclosed by the tangent to the curve $y=f(x)$ at the point where $x=1$ and the x- and y-axes. [6]

[J22/P1/Q10]

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Suggested Solution:

(a) Domain of $f^{-1}(x)$ is : $x<1, \quad x>1$ **(Ans).**

Alternatively, domain of $f^{-1}(x)$ is : $1<x<\infty, \quad -\infty<x<1$ **(Ans).**

(b) $f(x)=\dfrac{2x+1}{2x-1}$

Let $y=f(x)$

$\Rightarrow \quad y=\dfrac{2x+1}{2x-1}$

$\Rightarrow \quad y(2x-1)=2x+1$

$\Rightarrow \quad 2xy-y=2x+1$

$\Rightarrow \quad 2xy-2x=1+y \quad \Rightarrow \quad 2x(y-1)=1+y \quad \Rightarrow \quad x=\dfrac{1+y}{2(y-1)}$

Since, $f(x)=y \quad \Rightarrow \quad x=f^{-1}(y)$

$\Rightarrow \quad f^{-1}(y)=\dfrac{1+y}{2(y-1)}$,

$\therefore \quad f^{-1}(x)=\dfrac{1+x}{2(x-1)}$ **(Ans).**

(c) $f^{-1}(3)=\dfrac{1+3}{2(3-1)}=\dfrac{4}{2(2)}=1$

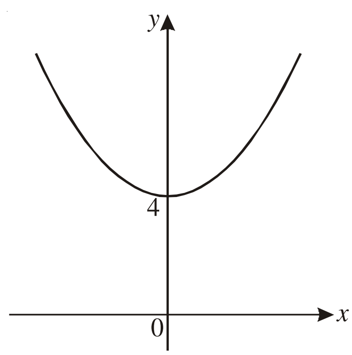
$\therefore \quad g f^{-1}(3)=g(1)$

$= (1)^2+4=5$ **(Ans).**

(d) $g(x)=x^2+4$

By drawing a sketch graph of $g(x)$ and using horizontal line test, we see that the horizontal line crosses the curve twice.

Thus, $g(x)$ cannot have an inverse because it is not a one-one function.



(e) $1+\dfrac{2}{2x-1}$

$=\dfrac{2x-1+2}{2x-1}=\dfrac{2x+1}{2x-1}$ **(Shown).**

Now, $f(x)=1+\dfrac{2}{2x-1}$

$\Rightarrow \quad f(x)=1+2(2x-1)^{-1}$

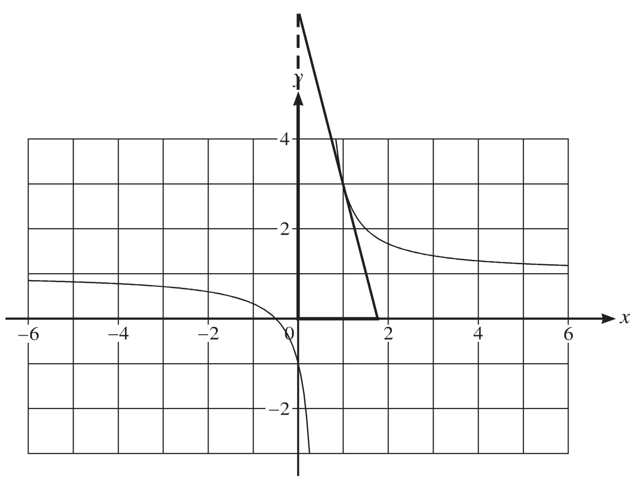
$f'(x)=0+2\left(-1(2x-1)^{-2}\times 2\right) = -\dfrac{4}{(2x-1)^2}$

At $x=1$, gradient of tangent is,

$f'(x)=-\dfrac{4}{(2(1)-1)^2}=-4$

Also, when $x=1$,

$f(x)=1+\dfrac{2}{2(1)-1}=1+2=3$



$\therefore$ Equation of tangent passing through $(1,\ 3)$ with gradient -4 is,

$y-3=-4(x-1)$

$\Rightarrow\ y-3=-4x+4\ \Rightarrow\ y+4x=7$

The tangent intersect the x-axis at $y=0,\ \Rightarrow\ 0+4x=7\ \Rightarrow\ x=\dfrac{7}{4}$

The tangent intersect the y-axis at $x=0,\ \Rightarrow\ y+4(0)=7\ \Rightarrow\ y=7$

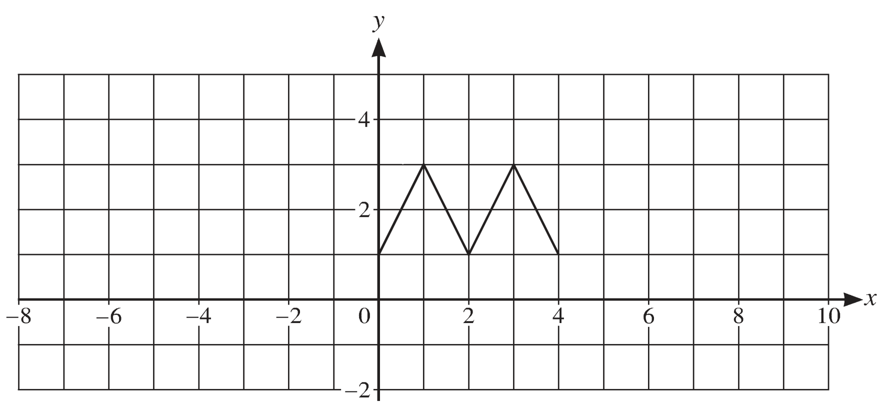
$\therefore$ Area of required triangle $=\dfrac{1}{2}\times\dfrac{7}{4}\times 7=\dfrac{49}{8}$ units^2 **(Ans).**

Question 35

The graph with equation $y=\mathrm{f}(x)$ is transformed to the graph with equation $y=\mathrm{g}(x)$ by a stretch in the x-direction with factor 0.5, followed by a translation of $\begin{pmatrix}0\\1\end{pmatrix}$.

(a) The diagram below shows the graph of $y=\mathrm{f}(x)$.

On the diagram sketch the graph of $y=\mathrm{g}(x)$. [3]

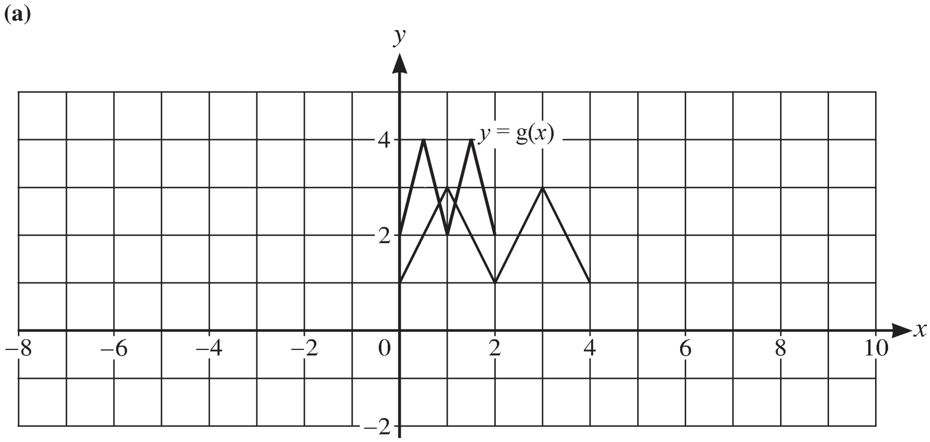


(b) Find an expression for $\mathrm{g}(x)$ in terms of $\mathrm{f}(x)$. [2]

[N22/P1/Q5]

Suggested Solution:

(a)



(b) $f(x) \to f\left(\frac{1}{0.5}x\right)$ due to horizontal stretch of factor 0.5

$f\left(\frac{1}{0.5}x\right) \to f\left(\frac{1}{0.5}x\right)+1$ due to translation of $\begin{pmatrix}0\\1\end{pmatrix}$

$\therefore\ g(x) = f\left(\frac{1}{0.5}x\right)+1$

$\Rightarrow\ g(x) = f(2x)+1$ **(Ans).**

Question 36

The equation of a curve is $y = 4x^2 + 20x + 6$.

(a) Express the equation in the form $y = a(x+b)^2 + c$, where a, b and c are constants. [3]

(b) Hence solve the equation $4x^2 + 20x + 6 = 45$. [3]

(c) Sketch the graph of $y = 4x^2 + 20x + 6$ showing the coordinates of the stationary point. You are not required to indicate where the curve crosses the x- and y-axes. [3]

[N22/P1/Q6]

Suggested Solution:

(a) $y = 4x^2 + 20x + 6$

Using completing the square method,

$$\begin{aligned} y &= 4(x^2+5x)+6 \\ &= 4\left[x^2 + 2(x)\left(\frac{5}{2}\right) + \left(\frac{5}{2}\right)^2 - \left(\frac{5}{2}\right)^2\right]+6 \\ &= 4\left[\left(x+\frac{5}{2}\right)^2 - \frac{25}{4}\right]+6 \\ &= 4\left(x+\frac{5}{2}\right)^2 - 25 + 6 = 4\left(x+\frac{5}{2}\right)^2 - 19 \quad \textbf{(Ans).} \end{aligned}$$

(b) $4x^2 + 20x + 6 = 45$

Using the result of part **(a)**,

$$4\left(x+\frac{5}{2}\right)^2 - 19 = 45$$

$$\Rightarrow\ 4\left(x+\frac{5}{2}\right)^2 = 64$$

$$\Rightarrow\ \left(x+\frac{5}{2}\right)^2 = 16 \ \Rightarrow\ \sqrt{\left(x+\frac{5}{2}\right)^2} = \pm\sqrt{16} \ \Rightarrow\ \left(x+\frac{5}{2}\right) = \pm 4$$

$$\Rightarrow\ x+\frac{5}{2} = 4 \quad \text{or} \quad x+\frac{5}{2} = -4$$

$$\Rightarrow\ x = 4-\frac{5}{2} \quad \text{or} \quad x = -4-\frac{5}{2}$$

$$\Rightarrow\ x = \frac{3}{2} \quad \text{or} \quad x = -\frac{13}{2}$$

$\therefore\ x = \frac{3}{2},\ -\frac{13}{2}$ **(Ans).**

(c) From the result of part **(a)**,

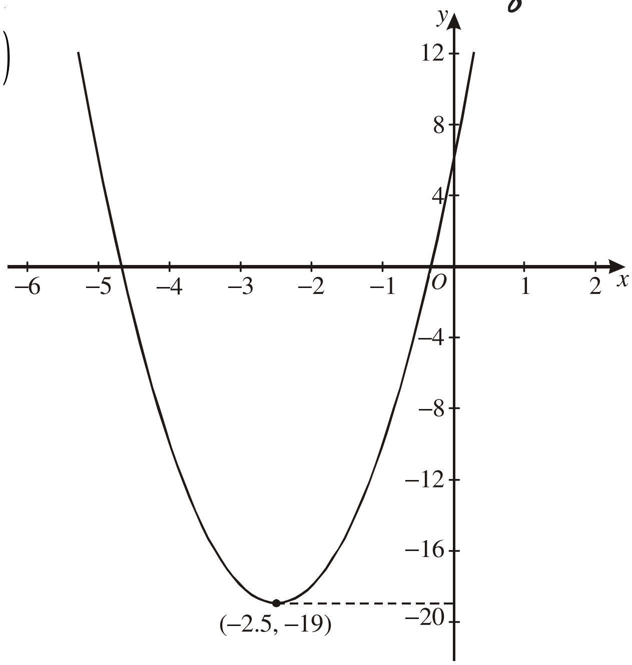
coordinates of the stationary point is $\left(-\frac{5}{2}, -19\right)$

For y-intercept, put $x = 0$,

$\Rightarrow \quad y = 4(0)^2 + 20(0) + 6 \quad \Rightarrow \quad y = 6$

The quadratic curve is opening upwards,

with axis of symmetry at $x = -\frac{5}{2}$



Question 37

Functions f and g are defined by

$$\mathrm{f}(x) = x + \frac{1}{x} \quad \text{for } x > 0,$$
$$\mathrm{g}(x) = ax + 1 \quad \text{for } x \in \mathbb{R},$$

where a is a constant.

(a) Find an expression for $\mathrm{gf}(x)$. [1]

(b) Given that $\mathrm{gf}(2) = 11$, find the value of a. [2]

(c) Given that the graph of $y = \mathrm{f}(x)$ has a minimum point when $x = 1$, explain whether or not f has an inverse. [1]

It is given instead that $a = 5$.

(d) Find and simplify an expression for $\mathrm{g}^{-1}\mathrm{f}(x)$. [3]

(e) Explain why the composite function fg cannot be formed. [1]

[N22/P1/Q9]

Suggested Solution:

(a) $\mathrm{gf}(x) = \mathrm{g}\left(x + \frac{1}{x}\right)$

$= a\left(x + \frac{1}{x}\right) + 1$ **(Ans).**

(b) $\mathrm{gf}(2) = 11$

Using the result of part **(a)**,

$a\left(2 + \frac{1}{2}\right) + 1 = 11$

$\Rightarrow \quad \frac{5}{2}a = 10 \quad \Rightarrow \quad a = \frac{20}{5} = 4$ **(Ans).**

(c) Given that the curve has a minimum point, it means that the curve is concaved up and so the function ceases to be a one-one function.
Therefore f does not have an inverse.

Alternatively,

Domain of $f(x)$ is $x > 0$. The curve has a minimum point at $x = 1$. This means that the curve lies on both sides of the line $x = 1$, and so does not pass the horizontal line test. Therefore f is not a one-one function and does not have an inverse.

(d) $g(x) = 5x + 1$

Let $g(x) = y$

$\Rightarrow \quad y = 5x+1 \quad \Rightarrow \quad 5x = y-1 \quad \Rightarrow \quad x = \dfrac{y-1}{5}$

$\Rightarrow \quad g^{-1}(y) = \dfrac{y-1}{5}, \qquad \therefore \quad g^{-1}(x) = \dfrac{x-1}{5}$

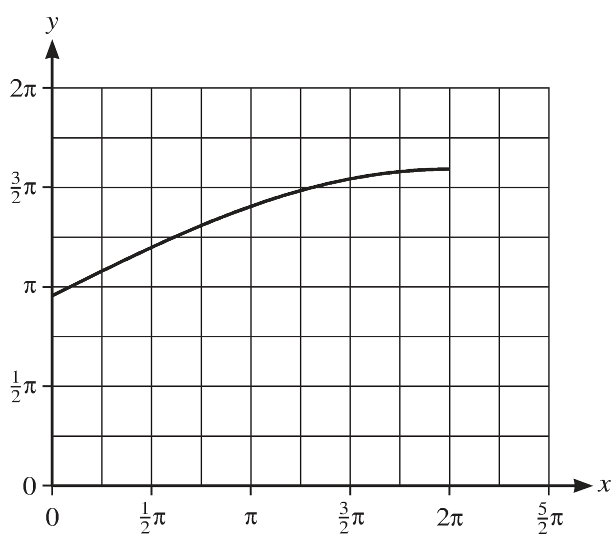
$$\begin{aligned} \text{Now,} \quad g^{-1}f(x) &= \frac{f(x)-1}{5} \\ &= \frac{x+\frac{1}{x}-1}{5} \\ &= \frac{\frac{x^2+1-x}{x}}{5} \\ &= \frac{x^2-x+1}{5x} \quad \textbf{(Ans).} \end{aligned}$$

(e) $fg(x)$ cannot be formed as the range of $g(x)$ is out of the domain of $f(x)$.

Or

$fg(x)$ cannot be formed as the domain of $f(x)$ does not include the whole of the range of $g(x)$.

Question 38



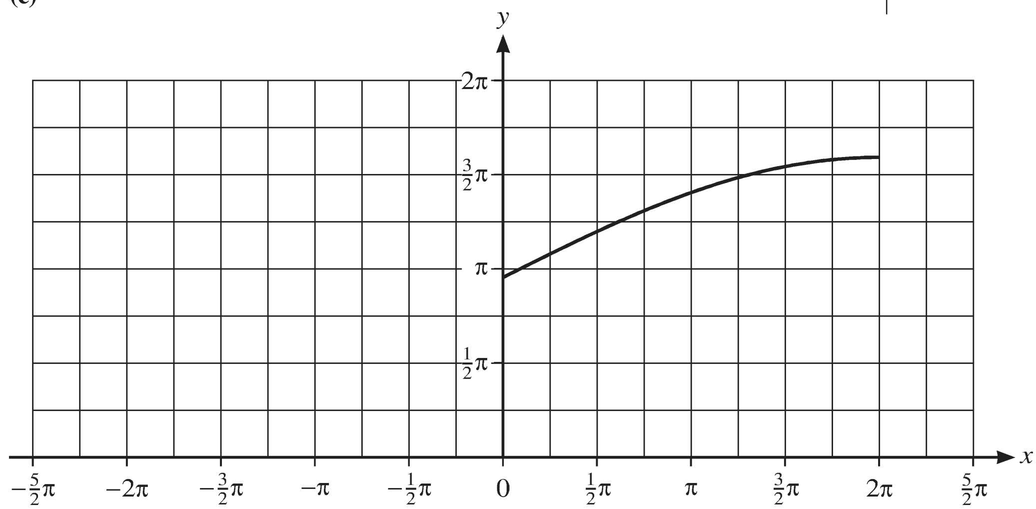
The diagram shows the graph of $y = f(x)$ where the function f is defined by

$$f(x) = 3 + 2\sin\frac{1}{4}x \quad \text{for} \quad 0 \le x \le 2\pi.$$

(a) On the diagram above, sketch the graph of $y = f^{-1}(x)$. [2]

(b) Find an expression for $f^{-1}(x)$. [2]

(c)



The diagram above shows part of the graph of the function $g(x) = 3 + 2\sin\frac{1}{4}x$ for $-2\pi \le x \le 2\pi$.

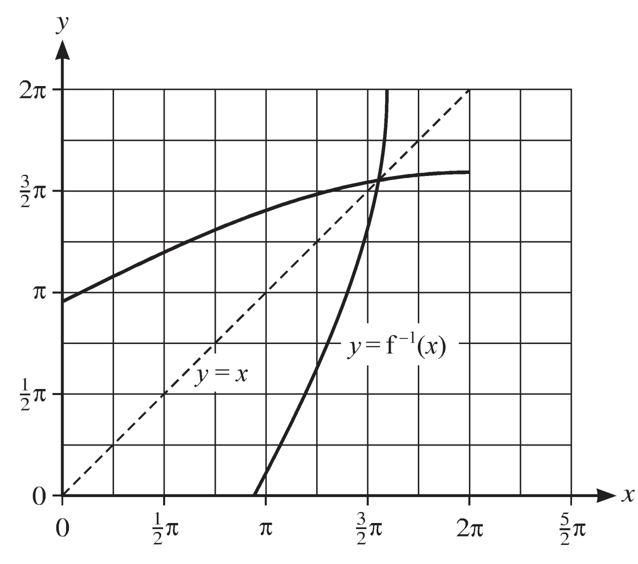
Complete the sketch of the graph of $g(x)$ on the diagram above and hence explain whether the function g has an inverse. [2]

(d) Describe fully a sequence of three transformations which can be combined to transform the graph of $y = \sin x$ for $0 \le x \le \frac{1}{2}\pi$ to the graph of $y = f(x)$, making clear the order in which the transformations are applied. [6]

[J23/P1/Q8]

Suggested Solution:

(a)



(b) $f(x) = 3 + 2\sin\frac{1}{4}x$

Let $y = f(x)$

$\Rightarrow \quad y = 3 + 2\sin\frac{x}{4}$

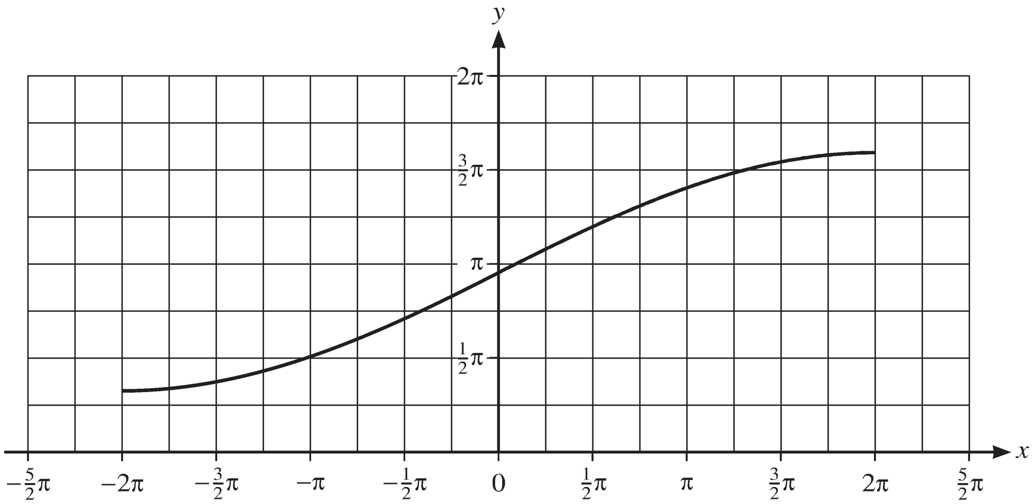
$\Rightarrow \quad 2\sin\frac{x}{4} = y - 3$

$\Rightarrow \quad \sin\frac{x}{4} = \frac{y-3}{2} \quad \Rightarrow \quad \frac{x}{4} = \sin^{-1}\left(\frac{y-3}{2}\right) \quad \Rightarrow \quad x = 4\sin^{-1}\left(\frac{y-3}{2}\right)$

Since $f(x) = y \quad \Rightarrow \quad x = f^{-1}(y)$

$\therefore \quad f^{-1}(y) = 4\sin^{-1}\left(\frac{y-3}{2}\right) \quad \Rightarrow \quad f^{-1}(x) = 4\sin^{-1}\left(\frac{x-3}{2}\right)$ **(Ans).**

(c)



Yes, the function has an inverse because it passes the horizontal line test and therefore is a (1 – 1) function.

(d) $\sin x \rightarrow \sin\frac{1}{4}x$: Stretch by scale factor 4 in x-direction

$\sin\frac{1}{4}x \rightarrow 2\sin\frac{1}{4}x$: Stretch by scale factor 2 in y-direction

$2\sin\frac{1}{4}x \rightarrow 3 + 2\sin\frac{1}{4}x$: Translation by column vector $\begin{pmatrix} 0 \\ 3 \end{pmatrix}$

Question 39

The equation of a curve is $y = x^2 - 8x + 5$.

(a) Find the coordinates of the minimum point of the curve. [2]

The curve is stretched by a factor of 2 parallel to the y-axis and then translated by $\begin{pmatrix} 4 \\ 1 \end{pmatrix}$.

(b) Find the coordinates of the minimum point of the transformed curve. [2]

(c) Find the equation of the transformed curve. Give the answer in the form $y = ax^2 + bx + c$, where a, b and c are integers to be found. [4]

[N23/P1/Q6]

Suggested Solution:

(a) $y = x^2 - 8x + 5$

$a = 1, \quad b = -8, \quad c = 5$

Coordinates of turning point

$$= \left(-\frac{b}{2a}, \ \frac{4ac - b^2}{4a}\right)$$

$$= \left(-\frac{-8}{2(1)}, \ \frac{4(1)(5) - (-8)^2}{4(1)}\right)$$

$$= \left(-4, \ \frac{-44}{4}\right) = (4, \ -11) \quad \textbf{(Ans).}$$

Alternative Method

$y = x^2 - 8x + 5$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 2x - 8$$

For minimum point, $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 0$

$\Rightarrow \quad 2x - 8 = 0 \quad \Rightarrow \quad 2x = 8 \quad \Rightarrow \quad x = 4$

Substitute $x = 4$ into equation of curve,

$$y = 4^2 - 8(4) + 5 \quad \Rightarrow \quad y = 16 - 32 + 5 = -11$$

$\therefore$ coordinates of min. point are, $(4, -11)$ **(Ans).**

(b) When curve is stretched by factor 2 along y-axis

The minimum point $(4, -11)$ moves to $(4, -22)$

During translation by $\begin{pmatrix} 4 \\ 1 \end{pmatrix}$.

$(4, -22)$ moves 4 units right and changes to $(8, -22)$

Finally, $(8, -22)$ moves 1 unit up and becomes $(8, -21)$ **(Ans).**

(c) $y = x^2 - 8x + 5$

Stretch along y-axis by factor 2

$\Rightarrow \quad y = 2(x^2 - 8x + 5)$

Translation by 4 units right

$\Rightarrow \quad y = 2\left((x-4)^2 - 8(x-4) + 5\right)$

Translation by 1 unit vetically up

$\Rightarrow \quad y = 2\left((x-4)^2 - 8(x-4) + 5\right) + 1$

$\Rightarrow \quad y = 2\left(x^2 - 8x + 16 - 8x + 32 + 5\right) + 1$

$\Rightarrow \quad y = 2\left(x^2 - 16x + 53\right) + 1$

$\Rightarrow \quad y = 2x^2 - 32x + 106 + 1$

$\Rightarrow \quad y = 2x^2 - 32x + 107$

$\therefore \quad a = 2, \quad b = -32, \quad c = 107$ **(Ans).**

Question 40

Functions f and g are defined by

$$\mathrm{f}(x) = (x+a)^2 - a \ \text{ for } \ x \le -a,$$
$$\mathrm{g}(x) = 2x - 1 \ \text{ for } \ x \in \mathbb{R},$$

where a is a positive constant.

(a) Find an expression for $\mathrm{f}^{-1}(x)$. [3]

(b) **(i)** State the domain of the function f^{-1}. [1]

(ii) State the range of the function f^{-1}. [1]

(c) Given that $a = \dfrac{7}{2}$, solve the equation $\mathrm{gf}(x) = 0$. [3]

[N23/P1/Q8]

Learning CORNER

Suggested Solution:

(a) $f(x) = (x+a)^2 - a$

Let $f(x) = y$

$\Rightarrow \; y = (x+a)^2 - a$

$\Rightarrow \; (x+a)^2 = y + a$

$\Rightarrow \; x + a = \pm\sqrt{y+a}$

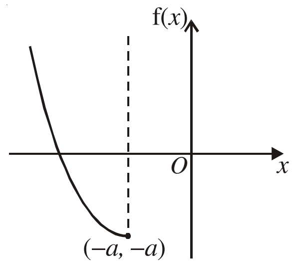
Since $f(x)$ is defined for $x \le -a$,

$\therefore \; x + a = -\sqrt{y+a} \;\Rightarrow\; x = -a - \sqrt{y+a}$

as $f(x) = y \;\Rightarrow\; x = f^{-1}(y)$

$\therefore \; f^{-1}(y) = -a - \sqrt{y+a} \;\Rightarrow\; f^{-1}(x) = -a - \sqrt{x+a}$ **(Ans).**

Domain of $f^{-1}(x)$ is the range of $f(x)$ and vice versa.



(b) **(i)** Range of $f(x)$ is: $f(x) \ge -a$

$\therefore$ Domain of $f^{-1}(x)$ is: $x \ge -a$ **(Ans).**

(ii) Range of $f^{-1}(x)$: $f^{-1}(x) \le -a$ **(Ans).**

(c) When $a = \frac{7}{2}$, $\quad f(x) = \left(x + \frac{7}{2}\right)^2 - \frac{7}{2}$, $\quad g(x) = 2x - 1$

$gf(x) = 0$

$\Rightarrow \; g\left[\left(x+\frac{7}{2}\right)^2 - \frac{7}{2}\right] = 0$

$\Rightarrow \; 2\left[\left(x+\frac{7}{2}\right)^2 - \frac{7}{2}\right] - 1 = 0$

$\Rightarrow \; 2\left(x+\frac{7}{2}\right)^2 - 7 - 1 = 0$

$\Rightarrow \; 2\left(x+\frac{7}{2}\right)^2 = 8$

$\Rightarrow \; \left(x+\frac{7}{2}\right)^2 = 4$

$\Rightarrow \; x + \frac{7}{2} = \pm 2$

$\therefore \; x + \frac{7}{2} = 2 \quad$ or $\quad x + \frac{7}{2} = -2$

$\Rightarrow \; x = 2 - \frac{7}{2} \quad$ or $\quad x = -2 - \frac{7}{2}$

$\Rightarrow \; x = -\frac{3}{2} \quad$ or $\quad x = -\frac{11}{2}$

$f(x)$ is defined for $x \le -\frac{7}{2}$. Therefore $f(x)$ does not exist at $x = -\frac{3}{2}$.

$\therefore \; x = -\frac{11}{2}$ **(Ans).**

Learning CORNER

Question 41

The curve $y = x^2$ is transformed to the curve $y = 4(x-3)^2 - 8$.
Describe fully a sequence of transformations that have been combined, making clear the order in which the transformations have been applied. [5]

[J24/P1/Q2]

Suggested Solution:

1. Translation $\begin{pmatrix}3\\0\end{pmatrix}$ along x-axis
2. Stretch by a factor of 4 parallel to y axis.
3. Translation of $\begin{pmatrix}0\\-8\end{pmatrix}$ along y-axis.

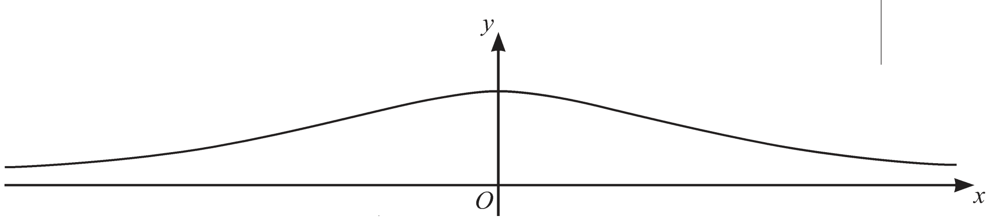
Note:
1. $(x-3)$ indicates a translation of 3 units to the right.
2. The coefficient 4 outside $(x-3)$ represent a vertical stretch of factor 4
3. -8 represents a translation of 8 units downwards.

Question 42

The function f is defined as follows:

$$f(x) = \sqrt{x} - 1 \text{ for } x > 1.$$

(a) Find an expression for $f^{-1}(x)$. [1]



The diagram shows the graph of $y = g(x)$ where $g(x) = \dfrac{1}{x^2+2}$ for $x \in \mathbb{R}$.

(b) State the range of g and explain whether g^{-1} exists. [2]

The function h is defined by $h(x) = \dfrac{1}{x^2+2}$ for $x \ge 0$.

(c) Solve the equation $hf(x) = f\left(\dfrac{25}{16}\right)$. Give your answer in the form $a + b\sqrt{c}$, where a, b and c are integers. [4]

[J24/P1/Q4]

Suggested Solution:

(a) $f(x) = \sqrt{x} - 1$, Let $y = f(x)$

$\Rightarrow \; y = \sqrt{x} - 1 \;\Rightarrow\; \sqrt{x} = y + 1 \;\Rightarrow\; x = (y+1)^2$

$\Rightarrow \; f^{-1}(y) = (y+1)^2$ (since, $x = f^{-1}(y)$)

$\therefore \; f^{-1}(x) = (x+1)^2$ **(Ans).**

(b) $g(x) = \dfrac{1}{x^2+2}$

For $x = 0$, $g(0) = \dfrac{1}{0^2+2} \Rightarrow g(0) = \dfrac{1}{2}$

$\therefore$ range of $g(x)$ is : $0 < g(x) \le \dfrac{1}{2}$ **(Ans).**

$g^{-1}(x)$ does not exist because $g(x)$ is not a one-one function. **(Ans).**

(c) $hf(x) = f\left(\dfrac{25}{16}\right)$

$\Rightarrow h\left(\sqrt{x}-1\right) = \sqrt{\dfrac{25}{16}} - 1$

$\Rightarrow \dfrac{1}{\left(\sqrt{x}-1\right)^2+2} = \dfrac{5}{4} - 1$

$\Rightarrow \dfrac{1}{\left(\sqrt{x}-1\right)^2+2} = \dfrac{1}{4}$

$\Rightarrow \left(\sqrt{x}-1\right)^2 + 2 = 4$

$\Rightarrow \left(\sqrt{x}-1\right)^2 = 2$

$\Rightarrow \sqrt{x} - 1 = \pm\sqrt{2}$

$\Rightarrow \sqrt{x} - 1 = \sqrt{2}$ or $\sqrt{x} - 1 = -\sqrt{2}$

$\Rightarrow \sqrt{x} = 1 + \sqrt{2}$ or $\sqrt{x} = 1 - \sqrt{2}$ (not valid since $1 - \sqrt{2}$ is negative)

$\therefore \sqrt{x} = 1 + \sqrt{2}$

$\Rightarrow x = \left(1+\sqrt{2}\right)^2 \Rightarrow x = 1 + 2\sqrt{2} + 2 \Rightarrow x = 3 + 2\sqrt{2}$ **(Ans).**

Question 43

A function f is such that $f'(x) = 6(2x-3)^2 - 6x$ for $x \in \mathbb{R}$.

Determine the set of values of x for which $f(x)$ is decreasing. [4]

[J24/P1/Q9(a)]

Suggested Solution:

$f'(x) = 6(2x-3)^2 - 6x$. Given that, $f(x)$ is decreasing,

$\Rightarrow f'(x) < 0$

$\Rightarrow 6(2x-3)^2 - 6x < 0$

$\Rightarrow 6\left((2x-3)^2 - x\right) < 0$

$\Rightarrow (2x-3)^2 - x < 0$

$\Rightarrow 4x^2 - 12x + 9 - x < 0$

$\Rightarrow 4x^2 - 13x + 9 < 0$

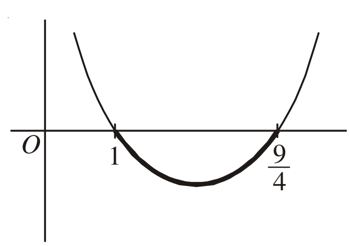
$\Rightarrow 4x^2 - 9x - 4x + 9 < 0$

$\Rightarrow x(4x-9) - 1(4x-9) < 0 \Rightarrow (x-1)(4x-9) < 0$

Critical values are, $x = 1$, $x = \dfrac{9}{4}$

$\therefore 1 < x < \dfrac{9}{4}$ **(Ans).**

- $f(x)$ is an increasing function if $f'(x) > 0$
- $f(x)$ is a decreasing function if $f'(x) < 0$.



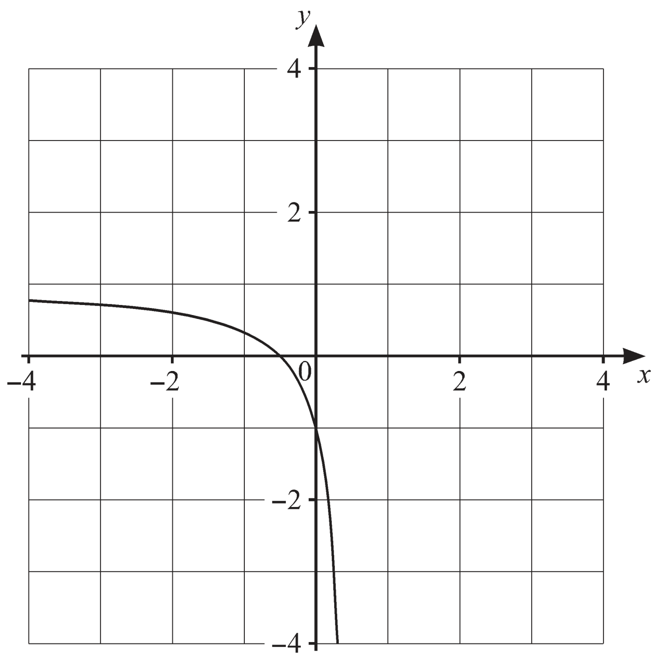
Learning CORNER

Question 44

The function f is defined by $\mathrm{f}(x)=\dfrac{2x+1}{2x-1}$ for $x<\dfrac{1}{2}$.

(a) **(i)** State the value of $\mathrm{f}(-1)$. [1]

(ii)



The diagram shows the graph of $y=\mathrm{f}(x)$. Sketch the graph of $y=\mathrm{f}^{-1}(x)$ on this diagram show any relevant mirror line. [2]

(iii) Find an expression for $\mathrm{f}^{-1}(x)$ and state the domain of the function f^{-1}. [4]

The function g is defined by $\mathrm{g}(x)=3x+2$ for $x\in\mathbb{R}$.

(b) Solve the equation $\mathrm{f}(x)=\mathrm{gf}\left(\dfrac{1}{4}\right)$. [3]

[N24/P1/Q5]

Suggested Solution:

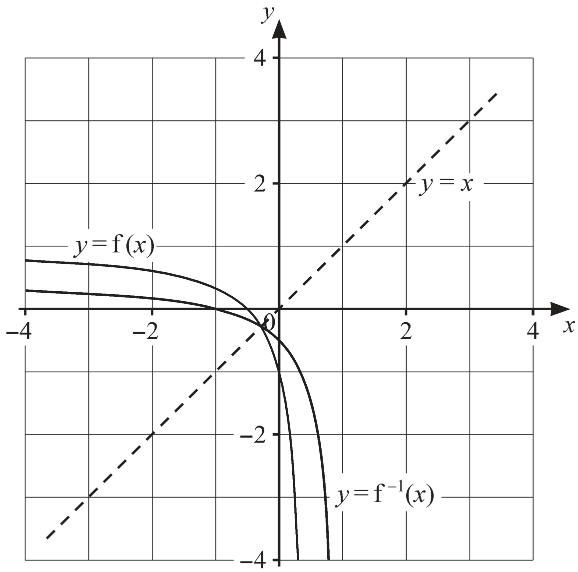
(a) **(i)** $\mathrm{f}(-1)=\dfrac{2(-1)+1}{2(-1)-1}$

$=\dfrac{-2+1}{-2-1}$

$=\dfrac{-1}{-3}=\dfrac{1}{3}$ **(Ans).**

Learning CORNER

(ii)



(iii) $f(x)=\dfrac{2x+1}{2x-1}$. Let $y=f(x)$

$\Rightarrow \quad y=\dfrac{2x+1}{2x-1}$

$\Rightarrow \quad 2xy-y=2x+1$

$\Rightarrow \quad 2xy-2x=y+1 \quad \Rightarrow \quad x(2y-2)=y+1 \quad \Rightarrow \quad x=\dfrac{y+1}{2y-2}$

Since, $f(x)=y \quad \Rightarrow \quad x=f^{-1}(y)$

$\therefore \quad f^{-1}(y)=\dfrac{y+1}{2y-2} \quad \Rightarrow \quad f^{-1}(x)=\dfrac{x+1}{2x-2}$ **(Ans).**

Domain of $f^{-1}(x)$ is, $x<1$ **(Ans).**

Domain of $f^{-1}(x)$ is the range of $f(x)$ and vice versa.

(b) $f\left(\dfrac{1}{4}\right)=\dfrac{2\left(\frac{1}{4}\right)+1}{2\left(\frac{1}{4}\right)-1}$

$=\dfrac{\frac{1}{2}+1}{\frac{1}{2}-1}=\dfrac{\frac{3}{2}}{-\frac{1}{2}}=\dfrac{3}{2}\times\left(-\dfrac{2}{1}\right)=-3$

Given that, $f(x)=gf\left(\dfrac{1}{4}\right)$

$\Rightarrow \quad f(x)=g(-3)$

$\Rightarrow \quad f(x)=3(-3)+2$

$\Rightarrow \quad f(x)=-7$

$\Rightarrow \quad x=f^{-1}(-7)$

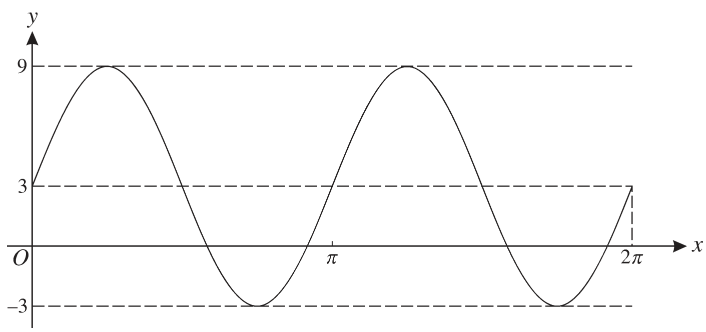
$\Rightarrow \quad x=\dfrac{-7+1}{2(-7)-2} \qquad \left(\text{since, } f^{-1}(x)=\dfrac{x+1}{2x-2}\right)$

$\Rightarrow \quad x=\dfrac{-6}{-14-2} \quad \Rightarrow \quad x=\dfrac{-6}{-16} \quad \Rightarrow \quad x=\dfrac{3}{8}$ **(Ans).**

Learning CORNER

Topic 5
Trigonometry

Question 1



The diagram shows the graph of $y = a\sin(bx) + c$ for $0 \le x \le 2\pi$.

(i) Find the values of a, b and c. [3]

(ii) Find the smallest value of x in the interval $0 \le x \le 2\pi$ for which $y = 0$. [3]

[J09/P1/Q4]

Suggested Solution:

(i) $y = a\sin(bx) + c$

a = amplitude of the curve

$\therefore$ from the graph, $a = 9 - 3 = 6$ **(Ans).**

As there are two cycles in the normal period of sin,

$\therefore$ $b = 2$ **(Ans).**

c indicates vertical shift in the graph from horizontal axis,

$\therefore$ $c = 3$ units **(Ans).**

(ii) Using the result of part (**i**), we have

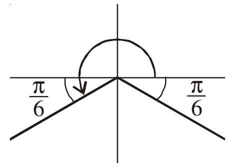
$y = 6\sin(2x) + 3$ for $0 \le x \le 2\pi$

at $y = 0$,

$$6\sin 2x + 3 = 0 \Rightarrow 6\sin 2x = -3 \Rightarrow \sin 2x = -\frac{1}{2}$$

basic angle, $\alpha = \dfrac{\pi}{6}$,

$\therefore$ for smallest value, $2x = \pi + \dfrac{\pi}{6} = \dfrac{7\pi}{6} \Rightarrow x = \dfrac{7\pi}{12}$ **(Ans).**



a: Indicates the amplitude of the curve which is the maximum or minimum range of the curve from the mean position.

b: Indicates the periodicity of the curve. We see that in the given figure there are two cycles in a normal period of sine.

c: Indicates the vertical shift in the graph from the horizontal axis, which in this case is 3 units.

Learning CORNER

Question 2

(i) Prove the identity $(\sin x + \cos x)(1 - \sin x \cos x) \equiv \sin^3 x + \cos^3 x$. [3]

(ii) Solve the equation $(\sin x + \cos x)(1 - \sin x \cos x) = 9\sin^3 x$ for $0° \le x \le 360°$. [3]

[N09/P1/Q5]

Suggested Solution:

(i) R.H.S. $= \sin^3 x + \cos^3 x$

$= (\sin x)^3 + (\cos x)^3$

$= (\sin x + \cos x)(\sin^2 x + \cos^2 x - \sin x \cos x)$

$= (\sin x + \cos x)(1 - \sin x \cos x) =$ L.H.S. **(Proved).**

Remember:

$(a)^3 + (b)^3$
$= (a+b)(a^2 + b^2 - ab)$

(ii) $(\sin x + \cos x)(1 - \sin x \cos x) = 9\sin^3 x$

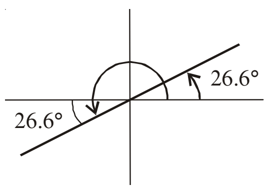
using the result in part (i), we have,

$\sin^3 x + \cos^3 x = 9\sin^3 x$

$\cos^3 x = 8\sin^3 x \Rightarrow \dfrac{\sin^3 x}{\cos^3 x} = \dfrac{1}{8} \Rightarrow \tan^3 x = \dfrac{1}{8} \Rightarrow \tan x = \dfrac{1}{2}$

basic angle, $\alpha = 26.6°$

$\therefore \quad x = 26.6°, \quad 206.6°$ **(Ans).**



Note that:
$\tan\theta$ is positive in 1st and 3rd quadrant.

Question 3

(i) Show that the equation

$$3(2\sin x - \cos x) = 2(\sin x - 3\cos x)$$

can be written in the form $\tan x = -\frac{3}{4}$. [2]

(ii) Solve the equation $3(2\sin x - \cos x) = 2(\sin x - 3\cos x)$, for $0° \le x \le 360°$. [2]

[J10/P1/Q1]

Suggested Solution:

(i) $3(2\sin x - \cos x) = 2(\sin x - 3\cos x)$

$6\sin x - 3\cos x = 2\sin x - 6\cos x$

$4\sin x = -3\cos x$

$\dfrac{\sin x}{\cos x} = -\dfrac{3}{4}$

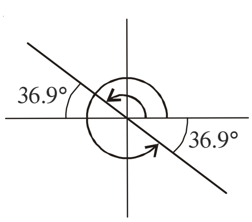
$\tan x = -\dfrac{3}{4}$ **(Shown).**

(ii) From part (i), $\tan x = -\dfrac{3}{4}$

basic Angle $\alpha = 36.9°$

$\therefore \quad x = 180° - 36.9° \;,\; 360° - 36.9°$

$= 143.1°, \;\; 323.1°$ **(Ans).**



Note that:

$\tan\theta$ is negative in II and IV quadrants.

Question 4

The function $f : x \mapsto 4 - 3\sin x$ is defined for the domain $0 \le x \le 2\pi$.

(i) Solve the equation f(*x*) = 2. [3]

(ii) Sketch the graph of *y* = f(*x*). [2]

(iii) Find the set of values of *k* for which the equation f(*x*) = *k* has no solution. [2]

The function $g : x \mapsto 4 - 3\sin x$ is defined for the domain $\frac{1}{2}\pi \le x \le A$.

(iv) State the largest value of *A* for which g has an inverse. [1]

(v) For this value of *A*, find the value of $g^{-1}(3)$. [2]

[J10/P1/Q11]

Suggested Solution:

(i) $f(x) = 4 - 3\sin x$ for $0 \le x \le 2\pi$

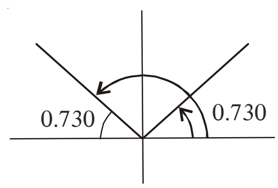
given that $f(x) = 2$

$\Rightarrow \quad 4 - 3\sin x = 2$

$\Rightarrow \quad 3\sin x = 2 \quad \Rightarrow \quad \sin x = \dfrac{2}{3}$

basic angle $\alpha = 0.730$ radian

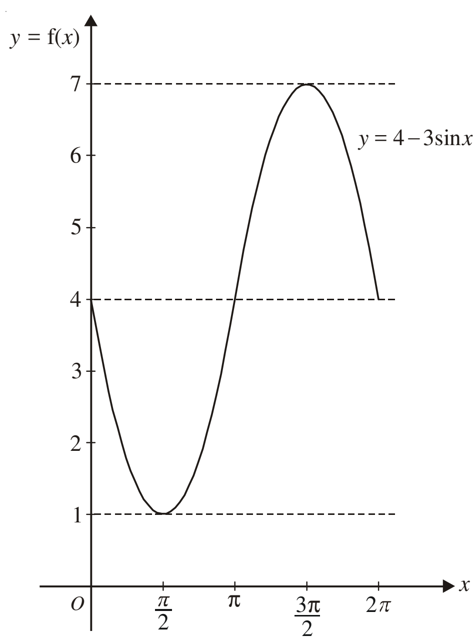
$\therefore \quad x = 0.730$ radian, 2.41 radians **(Ans).**



Note:
Change your calculator to radian mode.

(ii)

x	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
$y = f(x)$	4	1	4	7	4



(iii) From graph, we see that the curve has the range $1 \le y \le 7$.

Hence, the set of values of *k* for which f(*x*) = *k* has no solution are:

$k < 1$ or $k > 7$. **(Ans).**

(iv) $g(x) = 4 - 3\sin x, \quad \text{for} \quad \frac{1}{2}\pi \le x \le A$

From graph, the largest value of A in the given domain for which g has an inverse is $= \frac{3\pi}{2}$ **(Ans).**

(v) $g(x) = 4 - 3\sin x$

Let $g^{-1}(3) = x$

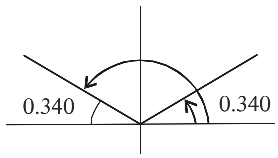
$\Rightarrow \quad g(x) = 3 \quad \Rightarrow \quad 4 - 3\sin x = 3 \quad \Rightarrow \quad 3\sin x = 1 \quad \Rightarrow \quad \sin x = \frac{1}{3}$

basic angle $\alpha = 0.340$ radians.

given range is $\frac{1}{2}\pi \le x \le \frac{3}{2}\pi$,

$\Rightarrow \quad x = \pi - 0.340 = 2.80$ radians.

$\therefore \quad g^{-1}(3) = 2.80$ **(Ans).**



Question 5

Prove the identity

$$\tan^2 x - \sin^2 x \equiv \tan^2 x \sin^2 x. \qquad [4]$$

[N10/P1/Q2]

Suggested Solution:

$$\begin{aligned}
\text{L.H.S} &= \tan^2 x - \sin^2 x \\
&= \frac{\sin^2 x}{\cos^2 x} - \frac{\sin^2 x}{1} \\
&= \frac{\sin^2 x - \sin^2 x \cos^2 x}{\cos^2 x} \\
&= \frac{\sin^2 x(1-\cos^2 x)}{\cos^2 x} \\
&= \frac{\sin^2 x(\sin^2 x)}{\cos^2 x} \\
&= \sin^2 x\left(\frac{\sin^2 x}{\cos^2 x}\right) = \sin^2 x \tan^2 x = \text{R.H.S} \quad \textbf{(Shown).}
\end{aligned}$$

Recall:

$1 - \sin^2\theta \equiv \cos^2\theta$

$1 - \cos^2\theta \equiv \sin^2\theta$

Question 6

(i) Prove the identity $\dfrac{\cos\theta}{\tan\theta(1-\sin\theta)} \equiv 1 + \dfrac{1}{\sin\theta}$. [3]

(ii) Hence solve the equation $\dfrac{\cos\theta}{\tan\theta(1-\sin\theta)} = 4$, for $0° \le \theta \le 360°$. [3]

[J11/P1/Q5]

Suggested Solution:

(i) $\text{L.H.S} = \dfrac{\cos\theta}{\tan\theta(1-\sin\theta)}$

$= \dfrac{\cos\theta}{\frac{\sin\theta}{\cos\theta}(1-\sin\theta)} = \dfrac{\cos^2\theta}{\sin\theta(1-\sin\theta)} = \dfrac{1-\sin^2\theta}{\sin\theta(1-\sin\theta)} = \dfrac{(1+\sin\theta)(1-\sin\theta)}{\sin\theta(1-\sin\theta)}$

$= \dfrac{1+\sin\theta}{\sin\theta} = \dfrac{1}{\sin\theta} + \dfrac{\sin\theta}{\sin\theta} = 1 + \dfrac{1}{\sin\theta} = \text{R.H.S}$ **(Proved).**

Recall:

$1-\sin^2\theta \equiv \cos^2\theta$

$1-\cos^2\theta \equiv \sin^2\theta$

(ii) $\dfrac{\cos\theta}{\tan\theta(1-\sin\theta)} = 4$

using the result of part **(i)**, we have,

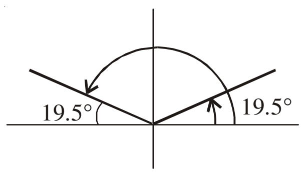
$1 + \dfrac{1}{\sin\theta} = 4 \quad \text{for } 0° \le \theta \le 360°$

$\Rightarrow \quad \dfrac{1}{\sin\theta} = 3$

$\Rightarrow \quad \sin\theta = \dfrac{1}{3}$

basic angle, $\alpha = 19.5°$

$\therefore \quad \theta = 19.5°, \quad 160.5°$ **(Ans).**



Note that:

$\sin\theta$ is positive in 1st and 2nd quadrants.

Question 7

The function f is such that $f(x) = 3 - 4\cos^k x$, for $0 \le x \le \pi$, where k is a constant.

(i) In the case where $k = 2$,

(a) find the range of f, [2]

(b) find the exact solutions of the equation $f(x) = 1$. [3]

(ii) In the case where $k = 1$,

(a) sketch the graph of $y = f(x)$, [2]

(b) state, with a reason, whether f has an inverse. [1]

[J11/P1/Q9]

Suggested Solution:

(i) **(a)** $f(x) = 3 - 4\cos^k x$

when $k = 2$, $\quad f(x) = 3 - 4\cos^2 x$

we know that,

$-1 \le \cos x \le 1$

$\Rightarrow \quad 0 \le \cos^2 x \le 1$

multiplying by -4

$0 \ge -4\cos^2 x \ge -4$

or $\quad -4 \le -4\cos^2 x \le 0$

adding 3

$-4 + 3 \le 3 - 4\cos^2 x \le 0 + 3$

$-1 \le 3 - 4\cos^2 x \le 3$

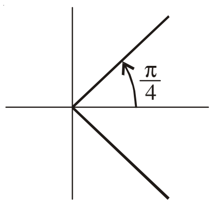
$\therefore \quad$ range of f is: $-1 \le f(x) \le 3$ **(Ans).**

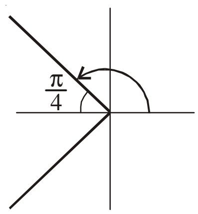
(b) $\mathrm{f}(x) = 1$

$\Rightarrow \quad 3 - 4\cos^2 x = 1 \quad \Rightarrow \quad 4\cos^2 x = 2 \quad \Rightarrow \quad \cos^2 x = \frac{1}{2} \quad \Rightarrow \quad \cos x = \pm\frac{1}{\sqrt{2}}$

$\Rightarrow \quad \cos x = \frac{1}{\sqrt{2}}$ or $\cos x = -\frac{1}{\sqrt{2}}$

basic angle $\alpha = \frac{\pi}{4}$ basic angle $\alpha = \frac{\pi}{4}$





$x = \frac{\pi}{4}$ $x = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$

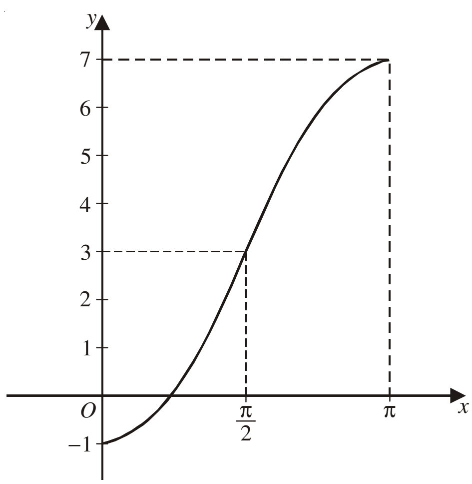
$\therefore \quad x = \frac{\pi}{4}, \ \frac{3\pi}{4}$ **(Ans).**

Note that:

$\cos\theta$ is positive in 1st and 4th quadrant.

$\cos\theta$ is negative in 2nd and 3rd quadrant.

(ii) (a) When $k = 1$, $\mathrm{f}(x) = 3 - 4\cos x$



(b) f has an inverse because f is a $1-1$ function.

Question 8

(i) Sketch, on the same diagram, the graphs of $y = \sin x$ and $y = \cos 2x$ for $0° \le x \le 180°$. [3]

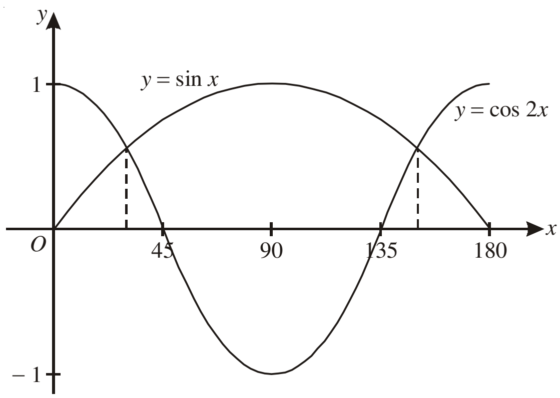
(ii) Verify that $x = 30°$ is a root of the equation $\sin x = \cos 2x$, and state the other root of this equation for which $0° \le x \le 180°$. [2]

(iii) Hence state the set of values of x, for $0° \le x \le 180°$, for which $\sin x < \cos 2x$. [2]

[N11/P1/Q5]

Suggested Solution:

(i)



(ii) $\sin x = \cos 2x$

at $x = 30°$,

$$\text{L.H.S.} = \sin 30° = \frac{1}{2}$$

$$\text{R.H.S.} = \cos 2(30°) = \cos 60° = \frac{1}{2}$$

$\therefore$ L.H.S. = R.H.S. **(Verified).**

symmeterically, from graph, $x = 150°$ is the other root, **(Ans).**

(iii) From graph, for $\sin x < \cos 2x$,

$0° \le x < 30°$ and $150° < x \le 180°$ **(Ans).**

Question 9

(i) Prove the identity $\tan x + \dfrac{1}{\tan x} \equiv \dfrac{1}{\sin x \cos x}$. [2]

(ii) Solve the equation $\dfrac{2}{\sin x \cos x} = 1 + 3\tan x$, for $0° \le x \le 180°$. [4]

[J12/P1/Q5]

Suggested Solution:

(i) $\text{L.H.S} = \tan x + \dfrac{1}{\tan x}$

$$= \frac{\sin x}{\cos x} + \frac{\cos x}{\sin x} = \frac{\sin^2 x + \cos^2 x}{\sin x \cos x} = \frac{1}{\sin x \cos x} = \text{R.H.S}$$ **(Proved).**

(ii) $\dfrac{2}{\sin x \cos x} = 1 + 3\tan x$, for $0° \le x \le 180°$

using the result of part (i),

$$2\left(\tan x + \frac{1}{\tan x}\right) = 1 + 3\tan x$$

$$\Rightarrow \quad 2\tan x + \frac{2}{\tan x} = 1 + 3\tan x$$

multiplying throughout by $\tan x$

$$2\tan^2 x + 2 = \tan x + 3\tan^2 x$$

$$\Rightarrow \quad \tan^2 x + \tan x - 2 = 0$$

$$\Rightarrow \quad \tan^2 x + 2\tan x - \tan x - 2 = 0$$

$$\Rightarrow \quad \tan x(\tan x + 2) - 1(\tan x + 2) = 0$$

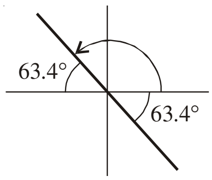
$$\Rightarrow \quad (\tan x + 2)(\tan x - 1) = 0$$

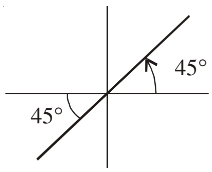
Learning CORNER

$\Rightarrow$ $\tan x + 2 = 0$ or $\tan x - 1 = 0$

$\tan x = -2$ or $\tan x = 1$

basic angle $\alpha = 63.43°$ basic angle $\alpha = 45°$





$\therefore$ $x = 116.6°$ $\therefore$ $x = 45°$

$\Rightarrow$ $x = 45°,\ 116.6°$ **(Ans).**

Note that:
$\tan\theta$ is positive in 1st and 3rd quadrant.

$\tan\theta$ is negative in 2nd and 4th quadrant.

Question 10

(i) Show that the equation $2\cos x = 3\tan x$ can be written as a quadratic equation in $\sin x$. [3]

(ii) Solve the equation $2\cos 2y = 3\tan 2y$, for $0° \le y \le 180°$. [4]

[N12/P1/Q6]

Suggested Solution:

(i) $2\cos x = 3\tan x$

$2\cos x = 3(\frac{\sin x}{\cos x})$

$2\cos^2 x = 3\sin x$

$2(1 - \sin^2 x) = 3\sin x$

$2 - 2\sin^2 x = 3\sin x \quad \Rightarrow \quad 2\sin^2 x + 3\sin x - 2 = 0$ **(Shown).**

(ii) $2\cos 2y = 3\tan 2y$

replacing $2y$ by x, we have,

$2\cos x = 3\tan x$

$\therefore$ using the result of part (i),

$2\sin^2 x + 3\sin x - 2 = 0$

$\Rightarrow$ $2\sin^2 x + 4\sin x - \sin x - 2 = 0$

$\Rightarrow$ $2\sin x(\sin x + 2) - 1(\sin x + 2) = 0$

$\Rightarrow$ $(\sin x + 2)(2\sin x - 1) = 0$

$\Rightarrow$ $\sin x + 2 = 0$ or $2\sin x - 1 = 0$

$\sin x = -2$ $\sin x = \frac{1}{2}$

(absurd & ignored) basic angle, $\alpha = 30°$

$\therefore$ $x = 30°,\ 150°$

$\Rightarrow$ $2y = 30°,\ 150°$

$\Rightarrow$ $y = 15°,\ 75°$ **(Ans).**

Question 36

(a) **(i)** By first expanding $(\cos\theta+\sin\theta)^2$, find the three solutions of the equation

$$(\cos\theta+\sin\theta)^2=1$$

for $0\le\theta\le\pi$. [3]

(ii) Hence verify that the only solutions of the equation $\cos\theta+\sin\theta=1$ for $0\le\theta\le\pi$ are 0 and $\frac{1}{2}\pi$. [2]

(b) Prove the identity $\dfrac{\sin\theta}{\cos\theta+\sin\theta}+\dfrac{1-\cos\theta}{\cos\theta-\sin\theta}\equiv\dfrac{\cos\theta+\sin\theta-1}{1-2\sin^2\theta}$. [3]

(c) Using the results of **(a)(ii)** and **(b)**, solve the equation

$$\frac{\sin\theta}{\cos\theta+\sin\theta}+\frac{1-\cos\theta}{\cos\theta-\sin\theta}=2(\cos\theta+\sin\theta-1).$$

for $0\le\theta\le\pi$. [3]

[J23/P1/Q7]

Suggested Solution:

(a) **(i)** $(\cos\theta+\sin\theta)^2=1$

$\Rightarrow\quad \cos^2\theta+\sin^2\theta+2\sin\theta\cos\theta=1$

$\Rightarrow\quad 1+2\sin\theta\cos\theta=1$

$\Rightarrow\quad 2\sin\theta\cos\theta=0$

$\Rightarrow\quad \sin\theta\cos\theta=0$

$\Rightarrow\quad \sin\theta=0 \quad$ or $\quad \cos\theta=0$

$\Rightarrow\quad \theta=0,\ \pi \quad$ or $\quad \theta=\frac{1}{2}\pi,\ \frac{3}{2}\pi$

$\therefore$ For the given range, $\theta=0,\ \frac{\pi}{2},\ \pi$ **(Ans).**

(ii) $\cos\theta+\sin\theta=1$

When $\theta=0$, $\cos 0+\sin 0=1$

$\Rightarrow\quad 1+0=1 \quad\Rightarrow\quad 1=1$ (satisfied)

When $\theta=\frac{\pi}{2}$, $\cos\frac{\pi}{2}+\sin\frac{\pi}{2}=1$

$\Rightarrow\quad 0+1=1 \quad\Rightarrow\quad 1=1$ (satisfied)

When $\theta=\pi$, $\cos\pi+\sin\pi=1$

$\Rightarrow\quad -1+0=1 \quad\Rightarrow\quad -1=1$ (does not satisfy)

Thus the only solutions of $\theta=0$ and $\frac{\pi}{2}$ **(Verified).**

(b) $\text{L.H.S} = \dfrac{\sin\theta}{\cos\theta + \sin\theta} + \dfrac{1 - \cos\theta}{\cos\theta - \sin\theta}$

$$= \frac{\sin\theta(\cos\theta - \sin\theta) + (1 - \cos\theta)(\cos\theta + \sin\theta)}{(\cos\theta + \sin\theta)(\cos\theta - \sin\theta)}$$

$$= \frac{\sin\theta\cos\theta - \sin^2\theta + \cos\theta + \sin\theta - \cos^2\theta - \sin\theta\cos\theta}{(\cos^2\theta - \sin^2\theta)}$$

$$= \frac{\cos\theta + \sin\theta - \sin^2\theta - \cos^2\theta}{1 - \sin^2\theta - \sin^2\theta}$$

$$= \frac{\cos\theta + \sin\theta - (\sin^2\theta + \cos^2\theta)}{1 - 2\sin^2\theta} = \frac{\cos\theta + \sin\theta - 1}{1 - 2\sin^2\theta} = \text{R.H.S} \quad \textbf{(Proved).}$$

Remember

$\sin^2\theta + \cos^2\theta = 1$

$\cos^2\theta = 1 - \sin^2\theta$

(c) $\dfrac{\sin\theta}{\cos\theta + \sin\theta} + \dfrac{1 - \cos\theta}{\cos\theta - \sin\theta} = 2(\cos\theta + \sin\theta - 1).$

Using the result of part **(b)**,

$$\frac{\cos\theta + \sin\theta - 1}{1 - 2\sin^2\theta} = 2(\cos\theta + \sin\theta - 1)$$

$$\Rightarrow \frac{\cos\theta + \sin\theta - 1}{1 - 2\sin^2\theta} - 2(\cos\theta + \sin\theta - 1) = 0$$

$$\Rightarrow (\cos\theta + \sin\theta - 1)\left(\frac{1}{1 - 2\sin^2\theta} - 2\right) = 0$$

$$\Rightarrow \cos\theta + \sin\theta - 1 = 0 \quad \text{or} \quad \frac{1}{1 - 2\sin^2\theta} - 2 = 0$$

$$\Rightarrow \cos\theta + \sin\theta = 1 \quad \text{or} \quad \frac{1}{1 - 2\sin^2\theta} = 2$$

$\therefore$ Using result of (a)(ii) $\quad \Rightarrow \quad 1 = 2 - 4\sin^2\theta$

$\theta = 0, \dfrac{\pi}{2} \quad \Rightarrow 4\sin^2\theta = 1$

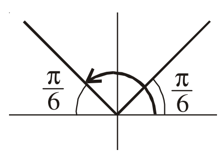
$$\Rightarrow \sin^2\theta = \frac{1}{4}$$

$$\Rightarrow \sin\theta = \frac{1}{2} \quad \text{or} \quad \sin\theta = -\frac{1}{2} \ \text{(out of given range)}$$

$$\text{basic angle} = \frac{\pi}{6}$$

$$\Rightarrow \theta = \frac{\pi}{6}, \ \pi - \frac{\pi}{6}$$

$$\therefore \ \theta = \frac{\pi}{6}, \ \frac{5\pi}{6}$$



$\therefore$ Solutions are, $\theta = 0, \dfrac{\pi}{6}, \dfrac{\pi}{2}, \dfrac{5\pi}{6}$ **(Ans).**

Question 37

Find the exact solution of the equation

$$\frac{1}{6}\pi + \tan^{-1}(4x) = -\cos^{-1}\left(\frac{1}{2}\sqrt{3}\right). \qquad [2]$$

[N23/P1/Q2]

Learning CORNER

Suggested Solution:

$$\frac{1}{6}\pi + \tan^{-1}(4x) = -\cos^{-1}\left(\frac{1}{2}\sqrt{3}\right)$$

$$\Rightarrow \frac{\pi}{6} + \tan^{-1}(4x) = -\frac{\pi}{6}$$

$$\Rightarrow \tan^{-1}(4x) = -\frac{\pi}{6} - \frac{\pi}{6}$$

$$\Rightarrow \tan^{-1}(4x) = -\frac{\pi}{3}$$

$$\Rightarrow 4x = \tan\left(-\frac{\pi}{3}\right)$$

$$\Rightarrow 4x = -\tan\left(\frac{\pi}{3}\right) \Rightarrow 4x = -\sqrt{3} \Rightarrow x = -\frac{\sqrt{3}}{4} \quad \textbf{(Ans).}$$

Note.

$\sin(-\theta) = -\sin\theta$

$\cos(-\theta) = \cos\theta$

$\tan(-\theta) = -\tan\theta$

Question 38

(a) Verify the identity $(2x-1)(4x^2+2x-1) \equiv 8x^3 - 4x + 1$. [1]

(b) Prove the identity $\dfrac{\tan^2\theta+1}{\tan^2\theta-1} \equiv \dfrac{1}{1-2\cos^2\theta}$. [3]

(c) Using the results of **(a)** and **(b)**, solve the equation

$$\frac{\tan^2\theta+1}{\tan^2\theta-1} = 4\cos\theta,$$

for $0° \le \theta \le 180°$. [5]

[N23/P1/Q7]

Suggested Solution:

(a) $\text{L.H.S.} = (2x-1)(4x^2+2x-1)$

$= 8x^3 + 4x^2 - 2x - 4x^2 - 2x + 1$

$= 8x^3 - 4x + 1 = \text{R.H.S.}$ **(Verified).**

(b) $\text{L.H.S.} = \dfrac{\tan^2\theta+1}{\tan^2\theta-1}$

$$= \frac{\dfrac{\sin^2\theta}{\cos^2\theta}+1}{\dfrac{\sin^2\theta}{\cos^2\theta}-1}$$

$$= \frac{\dfrac{\sin^2\theta+\cos^2\theta}{\cos^2\theta}}{\dfrac{\sin^2\theta-\cos^2\theta}{\cos^2\theta}}$$

$$= \frac{\sin^2\theta+\cos^2\theta}{\cos^2\theta} \times \frac{\cos^2\theta}{\sin^2\theta-\cos^2\theta}$$

$$= \frac{1}{\sin^2\theta-\cos^2\theta}$$

$$= \frac{1}{1-\cos^2\theta-\cos^2\theta} = \frac{1}{1-2\cos^2\theta} = \text{R.H.S.} \quad \textbf{(Proved).}$$

(c) $\dfrac{\tan^2\theta+1}{\tan^2\theta-1}=4\cos\theta$

Using the result of part **(b)**,

$\Rightarrow\ \dfrac{1}{1-2\cos^2\theta}=4\cos\theta$

$\Rightarrow\ 1=4\cos\theta(1-2\cos^2\theta)$

$\Rightarrow\ 1=4\cos\theta-8\cos^3\theta$

$\Rightarrow\ 8\cos^3\theta-4\cos\theta+1=0$

Using result of part **(a)**, and replacing x by $\cos\theta$, we have,

$\Rightarrow\ (2\cos\theta-1)(4\cos^2\theta+2\cos\theta-1)=0$

$\Rightarrow\ 2\cos\theta-1=0$ or $4\cos^2\theta+2\cos\theta-1=0$

$\Rightarrow\ \cos\theta=\dfrac{1}{2}$ or $\cos\theta=\dfrac{-2\pm\sqrt{2^2-4(4)(-1)}}{2(4)}$

$\Rightarrow\ \theta=60°$ or $\cos\theta=\dfrac{-2\pm\sqrt{20}}{8}$

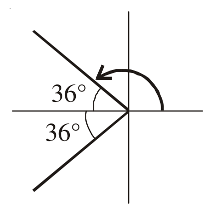
$\cos\theta=\dfrac{-2+\sqrt{20}}{8}$ or $\cos\theta=\dfrac{-2-\sqrt{20}}{8}$

$\cos\theta=0.3090$ or $\cos\theta=-0.8090$

basic angle $=72°$ or basic angle $=36°$

$\therefore\ \theta=72°$ $\therefore\ \theta=180°-36°$

$=144°$



$\therefore\ \theta=60°,\ 72°,\ 144°$ **(Ans).**

Question 39

(a) Show that the equation $\dfrac{7\tan\theta}{\cos\theta}+12=0$ can be expressed as

$12\sin^2\theta-7\sin\theta-12=0.$ [3]

(b) Hence solve the equation $\dfrac{7\tan\theta}{\cos\theta}+12=0$ for $0°\le\theta\le360°$. [3]

[J24/P1/Q3]

Suggested Solution:

(a) $\dfrac{7\tan\theta}{\cos\theta}+12=0$

$\Rightarrow\ \dfrac{7\tan\theta+12\cos\theta}{\cos\theta}=0$

$\Rightarrow\ 7\tan\theta+12\cos\theta=0$

$\Rightarrow\ 7\left(\dfrac{\sin\theta}{\cos\theta}\right)+12\cos\theta=0$

$\Rightarrow\ \dfrac{7\sin\theta+12\cos^2\theta}{\cos\theta}=0$

$\Rightarrow\ 7\sin\theta+12\cos^2\theta=0$

$\Rightarrow\ 7\sin\theta+12(1-\sin^2\theta)=0$

$\Rightarrow\ 7\sin\theta+12-12\sin^2\theta=0\ \Rightarrow\ 12\sin^2\theta-7\sin\theta-12=0$ **(Shown).**

Remember,
$\sin^2\theta+\cos^2\theta=1$
$\therefore\ \cos^2\theta=1-\sin^2\theta$

(b) $\dfrac{7\tan\theta}{\cos\theta}+12=0$

Using the result of part **(a)**,

$\Rightarrow\ 12\sin^2\theta-7\sin\theta-12=0$

$\Rightarrow\ 12\sin^2\theta-16\sin\theta+9\sin\theta-12=0$

$\Rightarrow\ 4\sin\theta(3\sin\theta-4)+3(3\sin\theta-4)=0$

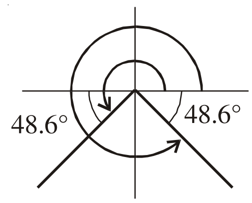
$\Rightarrow\ (3\sin\theta-4)(4\sin\theta+3)=0$

$\Rightarrow\ \sin\theta=\dfrac{4}{3}$(no solution) or $\sin\theta=-\dfrac{3}{4}$

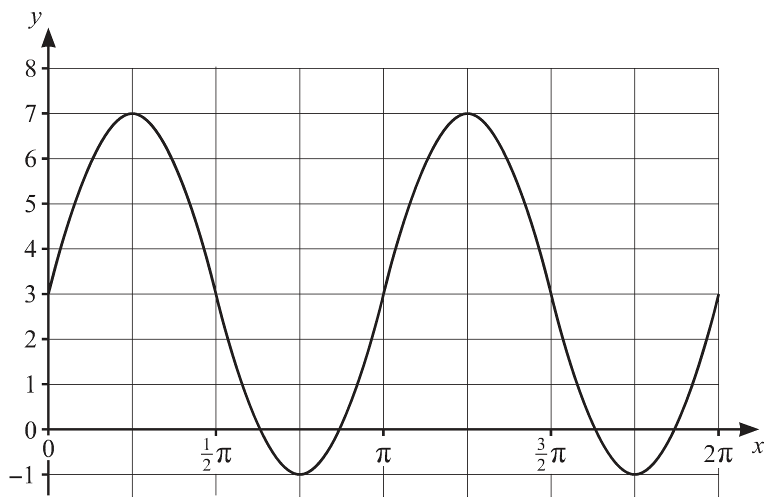
Basic angle, $\alpha=48.6°$

$\Rightarrow\ \theta=180°+48.6°,\ \ 360°-48.6°$

$\therefore\ \theta=228.6°,\ \ 311.4°$ **(Ans).**



Question 40



The diagram shows the curve with equation $y=a\sin(bx)+c$ for $0\le x\le 2\pi$, where a, b and c are positive constants.

(a) State the values of a, b and c. [3]

(b) For these values of a, b and c, determine the number of solutions in the interval $0\le x\le 2\pi$ for each of the following equations:

(i) $a\sin(bx)+c=7-x$ [1]

(ii) $a\sin(bx)+c=2\pi(x-1)$. [1]

[N24/P1/Q1]

Suggested Solution:

(a) $y=a\sin(bx)+c$

a is amplitude of the curve

$\therefore\ a=7-3=4$

From graph period $=\pi$

$\Rightarrow\ b=\dfrac{2\pi}{\pi}=2$

c = vertical translation of 3 units, $\Rightarrow\ c=3$

$\therefore\ a=4,\ \ b=2,\ \ c=3$ **(Ans).**

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(b) **(i)** $a\sin(bx)+c=7-x \quad \Rightarrow \quad y=7-x$

The line $y=7-x$, cuts the graph at 5 points

$\therefore$ Number of solutions = 5 **(Ans).**

(ii) $a\sin(bx)+c=2\pi(x-1)$

$\Rightarrow \quad y=2\pi(x-1)$

The line $y=2\pi(x-1)$ cuts the graph at one point

$\therefore$ Number of solutions = 1 **(Ans).**